PERTURBED RANDOM DIFFERENTIAL EQUATION

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Abstract: In this paper, the existence and attractivity results are proved for nonlinear first order perturbed ordinary random differential equation through random fixed point theorem of Dhage.

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1. DESCRIPTION OF THE PROBLEM

Given a measurable space (Ω, A) and a measurable function x : Ω → C(R+ , R), consider initially value problem of nonlinear first order ordinary perturbed random differential equation (in short PRDE)

$$\frac{d}{dt} x(t, \omega) - g(t, x(t, \omega), \omega) = f(t, x(t, \omega), \omega) \quad a.e. \ t \in R_+$$

$$x(0, \omega) = q_0(\omega)$$

(1.1)

for all \(\omega \in \Omega\), where, \(q_0 : \Omega \rightarrow R\) and \(f, g : R_+ \times R \times \Omega \rightarrow R\).

By a random solution of the RDE (1.1), I mean a measurable function \(x : \Omega \rightarrow AC(R_+, R)\) that satisfies space equation in (1.1), where, \(AC(R_+, R)\) is the space of absolutely continuous real- functions valued defined on \(R_+\).

The differential equation (1.1) is a random version of the deterministic differential equation

$$\frac{d}{dt} x(t) - g(t, x(t)) = f(t, x(t)) \quad a.e. \ t \in R_+$$

$$x(0) = q_0$$

(1.2)

Where \(f, g : R_+ \times R \rightarrow R\).

The differential equation (1.2) has been studied in Burton and Furu-mochi [2] for stability of solutions under some suitable growth condition via a fixed point theoretic technique of Krasnoselskii [8]. In this paper, I prove the existence of solution and attractivity results for the PRDE (1.1) via a random version of a hybrid fixed point theorem Dhage [5].

2. AUXILIARY RESULTS

I quote the following random fixed point theorem proving the main existence result.

Theorem 2.1 (Itoh [7]). Let \(X\) be a non-empty, closed convex bounded subset of the separable Banach space \(E\) and let \(Q : \Omega \times X \rightarrow X\) be a compact and continuous random operator. Then the random equation \(Q(\omega) x = x\) has a random solution, i. e there is a measurable function \(\xi : \Omega \rightarrow X\) such that \(Q(\omega) \xi(\omega) = \xi(\omega)\) for all \(\omega \in \Omega\).

The following theorem is used in the study of nonlinear discontinuous random differential equations.

Theorem 2.2(Carathe’dory). Let \(Q : \Omega \times E \rightarrow E\) be a mapping such that \(Q(\omega, x)\) is measurable for all \(x \in E\) and \(Q(\omega, \cdot)\) is continuous for all \(\omega \in \Omega\). Then the map \((\omega, x) \rightarrow Q(\omega, x)\) is jointly measurable.

3. EXISTENCE AND ATTRACTIVITY RESULTS

I have needed the following definition.

Definition 3.1.Afunction \(f : R_+ \times R \times \Omega \rightarrow R\) is called random Carathéodory if

i) the map \(\omega \rightarrow f(t, x, \omega)\) is measurable for all \(t \in R_+\) and \(x \in R\) and

ii) the map \(t \rightarrow f(t, x, \omega)\) is continuous for all \(x \in R\) and \(\omega \in \Omega\).
(ii) the map \((t, x) \rightarrow f(t, x, \omega)\) is jointly continuous for all \(\omega \in \Omega\).

Furthermore, a random Carathéodory function \(f : \mathbb{R}_+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}\) is called random \(L^1\)- Carathéodory, if there exists a function \(h \in L^1(\mathbb{R}_+, \mathbb{R})\) such that
\[
|f(t, x, \omega)| \leq h(t) \quad a.e. \ t \in \mathbb{R}_+
\]
for all \(\omega \in \Omega\) and \(x \in \mathbb{R}\). The function \(h\) is the growth function of \(f\) on \(\mathbb{R}_+ \times \mathbb{R} \times \Omega\).

I also quote the random version of the following random fixed point theorem of Dhage [5].

**Theorem 3.1.** Let \(X\) be a closed, convex and bounded subset of a Banach space \(E\) and let \(A : E \rightarrow E\) and \(B : X \rightarrow E\) be two operators such that:

(a) \(A\) is a nonlinear contraction,
(b) \(B\) is completely continuous, and
(c) \(Ax + B(\omega) y = x\) for all \(y \in X \Rightarrow x \in X\).

Then the operator equation \(Ax + Bx = x\) has a solution.

**Theorem 3.2.** Let \(X\) be a closed, convex and bounded subset of a separable Banach space \(E\) and let \(A : \Omega \times E \rightarrow E\) and \(B : \Omega \times X \rightarrow E\) be two operators such that for:

(a) \(A(\omega)\) is a nonlinear contraction,
(b) \(B(\omega)\) is completely continuous, and
(c) \(A(\omega) x + B(\omega) y = x\) for all \(y \in X \Rightarrow x \in X\).

Then the operator equation \(A(\omega) x + B(\omega) y = x\) has a random solution.

**Proof.** Let \(\omega \in \Omega\) be fixed and define a multi-valued map \(F : \Omega \rightarrow \mathcal{P}(E)\) by
\[
F(\omega) = x \in X : A(\omega)x + B(\omega)y = x.
\]

Clearly by Theorem 3.1, \(F(\omega)\) is non-empty for each \(\omega \in \Omega\). To finish, it is enough to prove that \(F\) is measurable on \(\Omega\). Let \(C\) be a closed subset of \(X\). Denote
\[
L(C) = \bigcap_{n=1}^{\infty} \bigcup_{x_i \in C_n} \{ \omega \in \Omega : \|x_i - (A(\omega)x_i + B(\omega)x_i)\| \leq 2/n \},
\]
where \(C_n = \{ x \in X : d(x, C) < 1/n \} \) and \(d(x, C) = \inf_{c \in C} d(x, c)\).

Clearly, \(L(C) \in \mathcal{A}\) shall prove that \(F^{-1}(C) = L(C)\). It is easy to prove that \(F^{-1}(C) \subseteq L(C)\). Now proceeding with the arguments similar to that given in the proof of Theorem 2.1 of Itoh [29], it is proved that \(F^{-1}(C) \subseteq L(C)\). As a result \(F^{-1}(C) = L(C)\). Hence \(F\) is measurable on \(\Omega\). Next we show that \(F(\omega)\) is closed for each \(\omega \in \Omega\). Let \(x_n\) be sequence of points in \(F(\omega)\) converging to a point \(x\). Since every nonlinear contraction random operator is continuous, I have
\[
A(\omega)x + B(\omega)x = \lim_{n \to \infty} A(x_n) + \lim_{n \to \infty} B(\omega)x_n = \lim_{n \to \infty} x_n = x
\]
as a result, \(x \in F(\omega)\) which thereby implies that \(F(\omega)\) is closed for each \(\omega \in \Omega\). Hence, the mapping \(F\) has closed values on \(\Omega\) into \(X\). Now an application of the measurable selector theorem of Kuratowski and Ryll-Nardzewskii (Itoh [7] and the references given therein) yields that \(F\) has a measurable selector, that is, there is measurable mapping \(\xi : \Omega \rightarrow X\) such that \(\xi(\omega) \in F(\omega)\) for each \(\omega \in \Omega\). Further by the definition of \(F(\omega)\) this implies that \(\xi(\omega) = A(\omega)\xi(\omega) + B(\omega)\xi(\omega)\), for all \(\omega \in \Omega\). This completes the proof.

Let \(X = \text{BC}(\mathbb{R}_+, \mathbb{R})\) and let \(M(\Omega , X)\) be the class of measurable \(X\)-valued function on \(\Omega\). I define the norm
\[
\| \cdot \|_\omega = \text{ess sup}_{\omega \in \Omega} \| x(\omega) \|.
\]
Two norms \(\| \cdot \|_\omega\) and \(\| \cdot \|\) are said to be comparable if there exists a constant \(c > 0\) such that
\[
c\| \cdot \| \leq \| \cdot \|_\omega \leq c\| \cdot \|.
\]

I need the following assumptions:
The function \( q_0 : \Omega \rightarrow R \) is measurable and bounded. Moreover,
\[
\text{ess sup}_{\omega \in \Omega} |q_0(\omega)| = c_1
\]
for some real number \( c_1 > 0 \).

The function \( f \) is random \( L^1 \)-Carathe`doory with growth function \( h \) on \( R_+ \).

Moreover,
\[
\lim_{t \to 0} e^{-K(t,0)} e^{(t,0)} h(s) ds = 0
\]
for all \( \omega \in \Omega \).

The function \( g : R_+ \times R \times \Omega \rightarrow R \) is measurable and bounded.

Moreover, we assume that
\[
G_{q_0} = \text{ess sup}_{\omega \in \Omega} g(0, q_0(\omega), \omega)
\]
and
\[
G_0 = \text{ess sup}_{\omega \in \Omega} \left( \sup_{t \in R_+} g(t, 0, \omega) \right)
\]

The norms \( \| \cdot \|_{L^1} \) and \( \| \cdot \|_{L^\infty} \) are comparable and there exists a measurable function \( l : R_+ \times \Omega \rightarrow R \) and a positive constant \( K \) such that for each \( \omega \in \Omega \),
\[
\left| g(t, x, \omega) - g(t, y, \omega) \right| \leq l(t, \omega) |x - y| / K + |x - y|
\]
for all \( t \in R_+ \) and \( x, y \in R \). Moreover, we assume that
\[
\lim_{t \to \infty} l(t, \omega) = 0 \quad \text{for all} \quad \omega \in \Omega,
\]
and
\[
\text{ess sup}_{\omega \in \Omega} \left( \sup_{t \in R_+} l(t, \omega) \right) = L.
\]

4. MAIN RESULT

Theorem 4.1. Assume that the hypothesis (A1)through (A4) hold. Further, if \( L \leq K \) then the PRDE (1.1) admits a random solution and solutions are globally uniformly attractive on \( R_+ \).

Proof. Now PRDE(1.1) is equivalent to the random equation
\[
x(t, \omega) = q_0(\omega) - g(0, q_0(\omega), \omega) \ e^t + g(t, x(t, \omega), \omega) + e^t \int_0^t e^{-s} f(s, x(s, \omega)) ds
\]
(4.1)
for all \( t \in R_+ \) and \( \omega \in \Omega \).

Set \( E = BC (R_+, R) \) define a closed ball \( B_r(0) \) centered at the origin of radius \( r = \|q_0\| + G_{q_0} + G_0 + L + W \). Define two operators \( A : \Omega \times E \rightarrow E \) and \( B : \Omega \times B^*_r(0) \rightarrow E \) by
\[
A(\omega) x(t) = g(t, x(t, \omega), \omega)
\]
(4.2)
\[
B(\omega) x(t) = q_0(\omega) - g(0, q_0(\omega), \omega) \ e^t + e^t \int_0^t e^{-s} f(s, x(s, \omega), \omega) ds
\]
for all \( t \in R_+ \) and \( \omega \in \Omega \).
From $(A_1)$ it follows that $A(\omega)$ is a random operator on $E$ in to itself. Here, I show that $B(\omega)$ is a random operator on $\Omega \times B_r(0)$ in to $B_r(0)$. By hypothesis $(A_1)$, the map $\omega \rightarrow f(t, x, \omega)$ is measurable by the Carathe’odory theorem. Since a continuous function is measurable, the map $t \rightarrow e^t$ is measurable and so the product $t \rightarrow e^t f(t, x(t, \omega), \omega)$ is measurable in $\omega$ for all $t \in R$ and $x \in R$. Since the integral is a limit of the finite sum of measurable functions, I have that the function $\omega \rightarrow \int_0^t e^{-s} f(s, x(s, \omega)) \, ds$ is measurable. Similarly, the map

$$\omega \rightarrow q_0(\omega) - g(0, q_0(\omega), \omega) e^t + g(t, x(t, \omega), \omega) + e^t \int_0^t e^{-s} f(s, x(s, \omega)) \, ds$$

is measurable for all $t \in R$. Consequently, the map $\omega \rightarrow B(\omega) x$ is measurable for all $x \in E$ and that $B(\omega)$ is a random operator on $\Omega \times B_r(0)$.

I will show that $A(\omega)$ and $B(\omega)$ satisfy all the conditions of Theorem 3.3. Firstly, I show that $A(\omega)$ is a nonlinear random contraction operator on $E$. Let $x, y \in E$. Then, by hypothesis $(A_2)$,

$$|A(\omega)x(t) - A(\omega)y(t)| \leq \left| g(t, x(t, \omega), \omega) - g(t, y(t, \omega), \omega) \right|$$

$$\leq \frac{l(t, \omega)|x(t, \omega) - y(t, \omega)|}{K + |x(t, \omega) - y(t, \omega)|} \leq \frac{L \|x - y\|}{K + \|x - y\|}$$

for all $\omega \in \Omega$. Taking supremum over $t$,

$$\|A(\omega)x - A(\omega)y\| \leq \frac{L \|x - y\|}{K + \|x - y\|}$$

for all $\omega \in \Omega$. Since $\phi(r) = \frac{Lr}{K + r} < r$ for $r > 0$, one has $A(\omega)$ is a nonlinear contraction random operator on $E$ in to itself.

Next, it can prove that $B(\omega)$ is a completely continuous random operator on $B_r(0)$ in to $E$. By hypothesis $(A_3)$,

$$\lim_{t \to r^-} w(t, \omega) = \lim_{t \to r^-} e^t \int_0^t e^{-s} h(s) \, ds = 0 ,$$

there is a real number $T > 0$ such that $w(t) < \frac{E}{4}$ for all $t \geq T$.

I show that the continuity of the random operator $B(\omega)$ in the following two cases:

Case I. Let $t \in 0, T$ and let $\{x_n\}$ be a sequence of points in $\tilde{B}_r(0)$ such that $x_n \to x$ as $n \to \infty$. Then, by the dominated convergence theorem,

$$\lim_{n \to \infty} \left( q_0(\omega) - g(0, q_0(\omega), \omega) e^t + g(t, x(t, \omega), \omega) + e^t \int_0^t e^{-s} f(s, x_n(s, \omega)) \, ds \right) = q_0(\omega) - g(0, q_0(\omega), \omega) e^t + g(t, x(t, \omega), \omega) + \lim_{n \to \infty} \left( e^t \int_0^t e^{-s} f(s, x_n(s, \omega)) \, ds \right)$$
for all $t \in [0, T]$ and $\omega \in \Omega$.

Case II. Suppose that $t \geq T$. Then we have
\[
\left| B(\omega) x_n(t) - B(\omega) x(t) \right| = \left| e^t \int_0^t e^{-s} f(s, x_n(s, \omega), \omega) \, ds - e^t \int_0^t e^{-s} f(s, x(s, \omega), \omega) \, ds \right|
\]
\[
\leq e^t \int_0^t |e^{-s} f(s, x_n(s, \omega), \omega)| \, ds
\]
\[
\leq e^t \int_0^t |e^{-s} f(s, x(s, \omega), \omega)| \, ds
\]
\[
\leq 2\omega(t) \varepsilon
\]
for all $t \geq T$ and $\omega \in \Omega$. Since $\varepsilon$ is arbitrary, one has $\lim_{n \to \infty} B(\omega) x_n(t) = B(\omega) x(t)$ for all $t \geq T$ and $\omega \in \Omega$. Now combining the Case I with Case II, I conclude that $B(\omega)$ is a pointwise continuous random operator on $\tilde{B}_r(0)$ into itself. Further, it is shown below that the family of functions $B(\omega) x_n$ is an equi-continuous set in $E$ for a fixed $\omega \in \Omega$. Hence, the above convergence is uniform on $R_+$ and consequently, $B(\omega)$ is a continuous random operator on $\tilde{B}_r(0)$ into itself.

Finally, assume that $x = A(\omega) x + B(\omega) y$ for all $y \in \tilde{B}_r(0)$. Then
\[
\left| x(t, \omega) - A(\omega) x(t) - B(\omega) y(t) \right|
\]
\[
\leq \left| g(t, x(t, \omega), \omega) + \left[ q(\omega) - g(0, q(\omega), \omega) \right] e^t \int_0^t e^{-s} f(s, x(s, \omega), \omega) \, ds \right|
\]
\[
\leq \frac{L}{K} \left| x(t, \omega) \right| + G_0 + \|q\| + G_q + W
\]
\[
\leq L + G_0 + \|q\| + G_q + W
\]
for all $t \in R_+$ and $\omega \in \Omega$. This shows that $x(\omega) \in B_r(0)$ for all $\omega \in \Omega$.

Thus, all the conditions of Theorem 4.3 are satisfied. Hence an application of it yields that the random operator equation $A(\omega) x + B(\omega) x = x$ has a random solution. As a result, the PRDE (1.1) has a random solution defined on $R_+$.

Let $x, y : \Omega \to B_r(0)$ be any two random solutions to the RDE (1.1) on $R_+$. Then, for each $\omega \in \Omega$,
\[
\left| x(t, \omega) - y(t, \omega) \right| \leq \left| g(t, x(t, \omega), \omega) - g(t, y(t, \omega), \omega) \right| + \left| e^t \int_0^t e^{-s} f(s, x(s, \omega), \omega) \, ds - e^t \int_0^t e^{-s} f(s, y(s, \omega), \omega) \, ds \right|
\]
\[ \frac{l(t,\omega)}{K + |x(t,\omega) - y(t,\omega)|} + e^\int_0^t e^{-s} |f_s, x(s,\omega), \omega)| \, ds + \]
\[ e^\int_0^t e^{-s} |f_s, x(s,\omega), \omega)| \, ds \]
\[ \leq l(t,\omega) + 2w(t,\omega) \]
for all \( t \in R_+ \).

Since \( \lim_{t \to \infty} l(t,\omega) = 0 \) and \( \lim_{t \to \infty} w(t,\omega) = 0 \) for each \( \omega \in \Omega \), there are real numbers \( T_1 > 0 \) and \( T_2 > 0 \) such that \( l(t,\omega) < \frac{\varepsilon}{2} \) for all \( t \geq T_1 \) and \( w(t,\omega) < \frac{\varepsilon}{4} \) for all \( t \geq T_2 \) and for all \( \omega \in \Omega \). Choose \( T = \max \{ T_1, T_2 \} \) then, \( |x(t,\omega) - y(t,\omega)| \leq \varepsilon \) for all \( t \geq T \) and for all \( \omega \in \Omega \). Hence all random solutions of the PRDE (1.1) are uniformly globally attractive on \( R_+ \).

REFERENCES


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