NEIGHBOURHOODS OF A CERTAIN SUBCLASS OF STARLIKE FUNCTIONS

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Abstract: The aim of this paper is to introduce the class $S^*_S(A, B, \beta)$ which is a subclass of $S^*(A, B, \beta)$ satisfying the condition

$$\frac{2zf''(z)}{f(z) - f(-z)} < \left(\frac{1 + Az}{1 + Bz}\right)^\beta, -1 \leq B < A \leq 1, 0 < B \leq 1, z \in E.$$ 

We study neighbourhoods of this class and also prove a necessary and sufficient condition in terms of convolutions for a function $f$ to be $S^*_S(A, B, \beta)$.

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Introduction: Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1)
which are analytic in the unit disk $E = \{z: |z| < 1\}$ Further let $S$ be the subclass of $A$ consisting of those functions that are univalent in $E$. Let $CV$ and $ST$ denote the subclasses of $S$ consisting of convex and starlike functions respectively.

If $f(z)$ and $g(z)$ are any two functions in $A$ such that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ then the convolution or Hadamard product of $f(z)$ and $g(z)$ denoted by $f \ast g$ is defined by

$$ (f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. $$

Clearly

$$ f(z) \ast \frac{z}{(1-z)^2} = zf'(z) \quad \text{and} \quad f(z) \ast \frac{z}{1-z^2} = \frac{f(z) - f'(z)}{2}. $$

The class of strongly starlike functions was introduced independently by Brannan and Kirwan [1] and Stankiewicz [4]

**Definition A:**

A function $f \in A$ is said to be strongly starlike of order $\alpha(0 < \alpha \leq 1)$ denoted by $STS(\alpha)$ if

$$ \left| \frac{zf'(z)}{f(z)} \right| \leq \frac{\alpha \pi}{2}, z \in E. \quad (2) $$
In terms of subordination $STS(\alpha)$ can be characterized as the class of functions $f$ satisfying the relation

$$\frac{zf''(z)}{f(z)} < \left(\frac{1+z}{1-z}\right)^\alpha$$

Recently Nalinakshi et al [2] generalize the class $STS(\alpha)$ by a class $S^* (A, B, \beta)$ if it satisfies

$$\frac{zf''(z)}{f(z)} < \left(\frac{1+AZ}{1+BZ}\right)^\beta, \quad -1 \leq B < A \leq 1, \quad 0 < \beta \leq 1, \quad z \in E. \quad (3)$$

The notion of $T - \delta$ neighbourhood was introduced by Sheil – Small and Silvia [3].

**Definition B:** For $\delta \geq 0$ and $T = \{T_n\}_{n=2}^{\infty}$ a sequence of non-negative reals, a $T - \delta$ neighbourhood of $f(z)$ analytic in $E$ is defined by

$$TN_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n : \sum_{n=2}^{\infty} T_n |a_n - b_n| \leq \delta \right\}. \quad (4)$$

We state a lemma which we need to establish our results.

**Lemma A [2]:** Let $f \in S^* (A, B, \beta)$. The set of values $\frac{zf''(z)}{f(z)}$ lie in an ellipse

$$\frac{(X-c)^2}{a^2} + \frac{Y^2}{b^2} = 1 \quad \text{where} \quad c = \frac{1}{2} \left[ \left(\frac{1+A}{1+B}\right)^\beta + \left(\frac{1-A}{1-B}\right)^\beta \right].$$

The major and minor axes
of the ellipse are given by $2a = \left(\frac{1+A}{1+B}\right)^\beta - \left(\frac{1-A}{1-B}\right)^\beta$, $2b = 2 \tan \beta \theta_0 \left(\frac{1-A^2}{1-B^2}\right)^{\beta/2}$

and $\theta_0 = \tan^{-1}\left(\frac{A-B}{1+AB}\right)$.

In this paper we introduce a new class of functions and study the properties of neighbourhoods of functions in this class which generalizes the recent results of Nalinakshi et al[2].

Now we define the class $S^*_{s'}(A,B,\beta)$ as follows.

**Definition 1:** Let $S^*_{s'}(A,B,\beta)$ be the class of all functions $h(z)$ defined on $E$ as,

$$h(z) = \frac{z}{(1-z)^2} - \left(\frac{c + \frac{a}{b} \sqrt{b^2 - t^2}}{1 - \left(c + \frac{a}{b} \sqrt{b^2 - t^2}ight)}\right) \left(\frac{z}{1-z^2}\right)$$

for some $t \in \mathbb{R}$

(5)

Where $a$, $b$ and $c$ are as defined in Lemma A

Now we give a characterization of the class $S^*_{s}(A,B,\beta)$ by means of convolution.

**Theorem 1:** $f \in S^*_{s}(A,B,\beta)$ if and only if $\frac{(f \ast h)(z)}{z} \neq 0$, $z \in E$, and for some

$h(z) \in S^*_{s'}(A,B,\beta)$. 
**Proof:** Let us first assume that $\frac{(f \ast h)(z)}{z} \neq 0$, and for some $h(z) \in S'_z(A,B,\beta)$ and $z \in E$ hence we have

\[
\frac{(f \ast h)(z)}{z} = \frac{f(z)^* \frac{z}{(1-z)^2} - \left(c + \frac{a}{b}\sqrt{b^2 - t^2} + it\right) f(z)^* \frac{z}{(1-z)^2}}{z \left(1 - \left(c + \frac{a}{b}\sqrt{b^2 - t^2} + it\right)\right)}
\]

\[
z f'(z) - \left(c + \frac{a}{b}\sqrt{b^2 - t^2} + it\right) \left(\frac{f(z) - f(-z)}{z}\right) \neq 0
\]

Equivalently

\[
\frac{2zf'(z)}{f(z) - f(-z)} \neq \left(c + \frac{a}{b}\sqrt{b^2 - t^2} + it\right)
\]

As $t$ varies $\left(c + \frac{a}{b}\sqrt{b^2 - t^2} + it\right)$ describes an ellipse.

At $z = 0$, $\frac{2zf'(z)}{f(z) - f(-z)} = 1$, hence $\frac{2zf'(z)}{f(z) - f(-z)}$ lies inside the ellipse, or $f \in S^*_z(A,B,\beta)$.

Conversely let $f \in S^*_z(A,B,\beta)$, then
\[
\frac{2zf'(z)}{f(z)-f(-z)} \neq \left( c + \frac{a}{b} \sqrt{b^2 - t^2} + it \right), \quad t \in \mathbb{R}.
\]

Equivalently

\[
f(z)^* \left\{ \frac{z}{(1-z)^2} - \left( c + \frac{a}{b} \sqrt{b^2 - t^2} + it \right) \left( \frac{z}{1-z^2} \right) \right\} \neq 0.
\]

Normalizing the function with in the brackets we get \( \frac{(f*h)(z)}{z} \neq 0 \) in \( E \) where \( h(z) \) is the function defined in (5).

To investigate the \( T-\delta \) neighbourhoods of functions belonging to the class \( S\beta^* (A,B,\beta) \), we need the following Lemmas.

**Lemma 1**: Let \( h(z) = z + \sum_{k=2}^{\infty} c_k z^k \) is in \( S\beta^* (A,B,\beta) \), then

\[
|c_k| \leq \frac{k}{\sigma \sqrt{1+\sigma^2 \delta_k}} \text{ where } \sigma = c + a - 1 \text{ and }
\]

\[
\delta_k = \begin{cases} 
0, & \text{if } k \text{ is even} \\
k^2, & \text{if } k \text{ is odd} 
\end{cases}
\]

**Proof**: Let \( h(z) \in S\beta^* (A,B,\beta) \) then for \( t \in \mathbb{R} \),
\[ h(z) = \frac{z}{(1-z)^2} \left( \frac{c+a}{b} \sqrt{b^2-t^2} + it \right) \frac{z}{(1-z^2)} \]

\[ = \frac{(z+2z^{2}+\ldots)-(c+a \sqrt{b^{2}-t^{2}}+it)(z+z^{3}+\ldots)}{1-(c+a \sqrt{b^{2}-t^{2}}+it)} \]

\[ = z + \sum_{k=2}^{\infty} c_k z^k \]

Then comparing the coefficients on either side we get,

\[ c_k = \begin{cases} 
\frac{k}{1-(c+a \sqrt{b^{2}-t^{2}}+it)}, & \text{when } k \text{ is even} \\
\frac{k-(c+a \sqrt{b^{2}-t^{2}}+it)}{1-(c+a \sqrt{b^{2}-t^{2}}+it)}, & \text{when } k \text{ is odd} 
\end{cases} \]

Hence when \( k \) is even

\[ |c_k| \leq \frac{k}{c+a-1} = \frac{k}{\sigma}, \quad \text{where } \sigma = c + a - 1. \]

when \( k \) is odd,
\[
\left| c_k \right|^2 \leq \frac{(k-1)(k+1-2c)}{(1-c^2)+a^2-2a(1-c)}
\]

\[
\leq k^2 \left[ 1 + \frac{1}{k^2 + \left( \frac{c+a}{c-a-1} \right)^2} \right]
\]

\[
\leq k^2 \left[ 1 + \frac{1}{(c+a-1)^2} \right] = k^2 \left[ \frac{(c+a-1)^2 + 1}{(c+a-1)^2} \right]
\]

that is

\[
\left| c_k \right| \leq \frac{k\sqrt{1+(c+a-1)^2}}{(c+a-1)} = \frac{k}{\sigma} \sqrt{1+\sigma^2}.
\]

As \( a + c - 1 = \left( \frac{1+A}{1+B} \right)^\beta - 1 \), we have

\[
\left| c_k \right| \leq \frac{k}{\left( \frac{1+A}{1+B} \right)^\beta}, \text{ when } k \text{ is even}
\]

and

\[
\left( \frac{1+A}{1+B} \right)^\beta - 1
\]
\[ |c_k| \leq \frac{k \left[ \left( \frac{1 + A}{1 + B} \right)^{2\beta} - 2 \left( \frac{1 + A}{1 + B} \right)^{\beta} + 2 \right]^{1/2}}{(1 + A)^{\beta} - 1}, \text{ when } k \text{ is odd} \]

**Lemma 2:** If for every \( \varepsilon \), \(|\varepsilon| < \delta < 1\), we have

\[ F_{\varepsilon}(z) = \frac{f(z) + \varepsilon z}{1 + \varepsilon} \in S_S^*(A, B, \beta) \] then for some

\[ h \in S_S^*(A, B, \beta), \left| \frac{(f * h)(z)}{z} \right| \geq \delta, z \in E. \]

**Proof:** Let \( F_{\varepsilon}(z) \in S_S^*(A, B, \beta) \) then by Theorem 1

\[ \left| \frac{(F_{\varepsilon} * h)(z)}{z} \right| \neq 0 \quad \forall h \in S_S^*(A, B, \beta), \text{and } z \in E. \]

Equivalently

\[ \frac{(f * h)(z) + \varepsilon z}{(1 + \varepsilon)z} \neq 0 \]

or

\[ \frac{(f * h)(z)}{z} \neq -\varepsilon \]

hence
\[
\frac{(f \ast h)(z)}{z} \geq \delta.
\]

**Theorem 2:** If for every \( \varepsilon, |\varepsilon| < \delta < 1 \), we have \( F_{\varepsilon}(z) \in S^*_S(A,B,\beta) \), then

\( TN_{\delta'}(f) \subset S^*_S(A,B,\beta) \), where \( \delta' = \delta/\gamma \) and

\[
\gamma = \sqrt{\left(\frac{1+A}{1+B}\right)^{2\beta} - 2\left(\frac{1+A}{1+B}\right)^\beta + 2} - \left(\frac{1+A}{1+B}\right)^\beta - 1.
\]

**Proof:** Let \( g(z) = z + \sum_{k=2}^{\infty} b_k z^k \) is in \( TN_{\delta'}(f) \) and \( h(z) \in S^*_S(A,B,\beta) \), then

\[
\left| \frac{(g \ast h)(z)}{z} \right| \geq \left| \frac{(f \ast h)(z)}{z} \right| - \left| \frac{(g - f) \ast h)(z)}{z} \right|
\]

\[
\geq \delta - \frac{1}{z} \sum_{k=2}^{\infty} |a_k - b_k| z^k
\]

\[
> \delta - \sum_{k=2}^{\infty} \gamma k |a_k - b_k| = \delta - \gamma \delta' = 0 \text{ for } \delta' = \frac{\delta}{\gamma}.
\]
References:


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