Weak $L^p[0, 1] \setminus L^p[0, 1]$ is lineable

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Abstract: Weak $L^p$ spaces are function spaces that are closed to $L^p$ spaces, but somehow larger. The question that we are going to partially answer in this paper, is that how much it is larger. Actually we prove that Weak $L^p[0, 1] \setminus L^p[0, 1] \cup \{0\}$ contains an infinite dimensional vector space.

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1. INTRODUCTION

Weak $L^p$ spaces are function spaces that are closely related to $L^p$ spaces. We do not know the exact origin of Weak $L^p$ spaces. The Book by Colin Bennett and Robert Sharpley [?] contains a good presentation of Weak $L^p$ but from the point of view of rearrangement function. In the present paper we first study the Weak $L^p$ space from the point of view of distribution function. This circumstance motivated us to undertake a preparation of the present paper containing a detailed exposition of these function spaces. Then we proceed to the main theorem that proves the existence of an infinitely generated vector space in Weak $L^p[0, 1] \setminus L^p[0, 1] \cup \{0\}$. A subset $M$ of a vector space $X$ is called lineable in $X$ if $M \cup \{0\}$ contains an infinite dimensional vector space. In this setting, we are going to show that Weak $L^p[0, 1] \setminus L^p[0, 1]$ is lineable in Weak $L^p[0, 1]$.

The origin of lineability is due to Gurariy ([?, ?]) that showed that there exists an infinite dimensional linear space such that every non-zero element of which is a continuous nowhere differentiable function on $C[0; 1]$. Many examples of vector spaces of functions on $\mathbb{R}$ or $\mathbb{C}$ enjoying certain special properties have been constructed in the recent...
years. More recently, many authors got interested in this subject and gave a wide range of examples. For more results on lineability we refer the reader to [?].

Throughout this paper \((X, \Omega, \mu)\) is a measure space and \(\mathbb{F}\) is \(\mathbb{R}\) or \(\mathbb{C}\).

**Definition 1.1.** For \(f : X \to \mathbb{F}\) a measurable function on \(X\), the distribution function of \(f\) is the function \(D_f\) defined on \([0, \infty)\) as follows

\[
D_f(\lambda) := \mu \left( \{ x \in X : |f(x)| > \lambda \} \right).
\]

The distribution function \(D_f\) provides information about the size of \(f\) but not about the behavior of \(f\) itself near any given point. For instance, a function on \(\mathbb{R}^n\) and each of its translates have the same distribution function. It follows from ?? that \(D_f\) is a decreasing function of \(\lambda\) (not necessarily strictly).

Let \((X, \Omega, \mu)\) be a measurable space and \(f\) and \(g\) be a measurable functions on \(X\) then \(D_f\) enjoys the following properties.

1. \(|g| \leq |f| \mu - a.e.\) implies that \(D_g \leq D_f\);
2. \(D_{cf}(\lambda) = D_f \left( \frac{\lambda}{|c|} \right)\) for all \(c \in \mathbb{C} \setminus \{0\}\) and \(\lambda \in [0, \infty)\);
3. \(D_{f+g}(\lambda_1 + \lambda_2) \leq D_f(\lambda_1) + D_g(\lambda_2)\) for all \(\lambda_1, \lambda_2 \in [0, \infty)\);
4. \(D_{fg}(\lambda_1 \lambda_2) \leq D_f(\lambda_1) + D_g(\lambda_2)\) for all \(\lambda_1, \lambda_2 \in [0, \infty)\);

For more details on distribution function see [?] and [?].

Next, Let \((X, \Omega, \mu)\) be a measurable space, for \(0 < p < \infty\), we consider

\[
\text{Weak } L^p := \left\{ f : X \to \mathbb{F} : \exists c > 0 \forall \lambda > 0, D_f(\lambda) \leq \left( \frac{c}{\lambda} \right)^p \right\}.
\]

We will use the notation \(L^p_w\) to show Weak \(L^p\). Observe that \(L^\infty_w = L^\infty\).

**Proposition 1.2.** Let \(f \in L^p_w\) with \(0 < p < \infty\). Then

\[
||f||_{L^p_w} := \inf \left\{ c > 0 : D_f(\lambda) \leq \left( \frac{c}{\lambda} \right)^p \text{ for all } \lambda \in (0, \infty) \right\}
\]

\[
= \left( \sup_{\lambda > 0} \lambda^p D_f(\lambda) \right)^{\frac{1}{p}}
\]

\[
= \sup_{\lambda > 0} \lambda \{ D_f(\lambda) \}^{\frac{1}{p}}.
\]

**Proof.** Let us define

\[
\alpha = \inf \left\{ c > 0 : D_f(\lambda) \leq \left( \frac{c}{\lambda} \right)^p \text{ for all } \lambda \in (0, \infty) \right\},
\]

and

\[
\beta = \left( \sup_{\lambda > 0} \lambda^p D_f(\lambda) \right)^{\frac{1}{p}}.
\]
Since $f \in L^p_w$, so there exists $c > 0$ such that
\[ D_f(\lambda) \leq \left( \frac{c}{\lambda} \right)^p, \; \lambda \in (0, \infty). \]
Therefore
\[ \{ c > 0 : D_f(\lambda) \leq \left( \frac{c}{\lambda} \right)^p \text{ for all } \lambda \in (0, \infty) \} \neq \emptyset. \]
On the other hand
\[ \lambda^p D_f(\lambda) \leq \beta^p, \]
thus $\{ \lambda^p D_f(\lambda) : \lambda > 0 \}$ is bounded above by $\beta^p$ and $\beta \in \mathbb{R}$.
Therefore
\[ (2) \quad \alpha = \inf \left\{ c > 0 : D_f(\lambda) \leq \left( \frac{c}{\lambda} \right)^p \text{ for all } \lambda \in (0, \infty) \right\} \leq \beta. \]
Now, let $\varepsilon > 0$, then there exists $c > 0$ such that $\alpha \leq c \leq \alpha + c$, and
\[ D_f(\lambda) \leq \left( \frac{c}{\lambda} \right)^p \text{ for all } \lambda \in (0, \infty) \]
and thus
\[ \lambda^p D_f(\lambda) \leq c^p \leq (\alpha + \varepsilon)^p, \; \lambda > 0. \]
then
\[ \sup_{\lambda > 0} \lambda^p D_f(\lambda) \leq (\alpha + \varepsilon)^p. \]
Therefore
\[ \left( \sup_{\lambda > 0} \lambda^p D_f(\lambda) \right)^{\frac{1}{p}} \leq \alpha + \varepsilon. \]
Since $\varepsilon$ is arbitrary so $\beta \leq \alpha$. This completes the proof. \qed

By this norm, we can redefine $L^p_w$ spaces in the form of $L^p$ spaces.

**Definition 1.3.** Let $(X, \Omega, \mu)$ be a measure space. For $0 < p < \infty$ the space $L^p_w$ is defined as the set of all $\mu$-measurable $\mathbb{F}$-valued functions $f$ such that
\[ \inf \left\{ c > 0 : D_f(\lambda) \leq \left( \frac{c}{\lambda} \right)^p \text{ for all } \lambda \in (0, \infty) \right\} < \infty. \]
Two functions in $L^p_w$ will be considered equal if they are equal $\mu$-a. e.

For $0 < p < \infty$, $L^p_w$ is larger than $L^p$. We obtain this result in the next proposition and remark.

**Proposition 1.4.** For any $0 < p < \infty$, $L^p \subset L^p_w$ and for any $f \in L^p$ we have
\[ ||f||_{L^p_w} \leq ||f||_{L^p}. \]
Proof. If $f \in L^p$, then for any $\lambda > 0$ we have
\[ \lambda^p \mu \left( \{ x \in X : |f(x)| > \lambda \} \right) \leq \int_{|f|>\lambda} |f|^p d\mu \leq \int_X |f|^p d\mu = \|f\|_{L^p}^p. \]
Therefore
\[ \mu \left( \{ x \in X : |f(x)| > \lambda \} \right) \leq \left( \frac{\|f\|_{L^p}}{\lambda} \right)^p \]
Hence $f \in L^p_w$, which means that $L_p \subset L^p_w$.

Next from ?? we have
\[ \left( \sup_{\lambda>0} \lambda^p D_f(\lambda) \right)^{\frac{1}{p}} \leq \|f\|_{L^p}. \]
This shows that
\[ \|f\|_{L^p_w} \leq \|f\|_{L^p}. \]

Remark 1.5. The inclusion in the previous proposition is strict. Indeed let $f(x) = x^{-\frac{1}{p}}$ on $(0, \infty)$ with the Lebesgue measure. For any $\lambda > 0$ we have
\[ m \left( \left\{ x \in (0, \infty) : \frac{1}{|x|^\frac{1}{p}} > \lambda \right\} \right) = m \left( \left\{ x \in (0, \infty) : |x| < \frac{1}{\lambda^p} \right\} \right) = 2\lambda^{-p}. \]
Thus $f \in L^p_w(0, \infty)$ but
\[ \int_0^\infty \left( \frac{1}{x^p} \right)^p dx = \int_0^\infty \frac{dx}{x} \to \infty. \]
So $f \notin L^p(0, \infty)$.

2. Main Result

In this section we prove that $L^p_w[0, 1] \setminus L^p[0, 1] \cup \{0\}$ contains an infinite dimensional vector space. First we present two facts about norm of $L^p_w$.

**Proposition 2.1.** Let $f, g \in L^p_w$. Then
\[ (1) \quad \| cf \|_{L^p_w} = |c| \| f \|_{L^p_w}, \text{ for any constant } c, \]
\[ (2) \quad \| f + g \|_{L^p_w} \leq 2 \left( \| f \|_{L^p_w}^p + \| g \|_{L^p_w}^p \right)^{\frac{1}{p}}. \]
Proof. (1) For $c > 0$ we have
\[
\mu(\{x \in X : |cf(x)| > \lambda\}) = \mu\left(\left\{ x \in X : |f(x)| > \frac{\lambda}{c}\right\}\right),
\]
thus
\[
D_{cf}(\lambda) = D_f\left(\frac{\lambda}{c}\right).
\]
Therefore
\[
||cf||_{L^p_w} = \left(\sup_{\lambda > 0} \lambda^p D_{cf}(\lambda)\right)^{\frac{1}{p}} = \left(\sup_{\lambda > 0} \lambda^p D_f\left(\frac{\lambda}{c}\right)\right)^{\frac{1}{p}} = \left(\sup_{\omega > 0} c^p \omega^p D_f(\omega)\right)^{\frac{1}{p}}.
\]
So
\[
||cf||_{L^p_w} = c||f||_{L^p_w}.
\]
(2) Note that
\[
\{x \in X : |f(x) + g(x)| > \lambda\}
\subseteq \left\{ x \in X : |f(x)| > \frac{\lambda}{2}\right\} \cup \left\{ x \in X : |g(x)| > \frac{\lambda}{2}\right\}.
\]
Hence
\[
\mu(\{x \in X : |f(x) + g(x)| > \lambda\})
\leq \mu\left(\left\{ x \in X : |f(x)| > \frac{\lambda}{2}\right\}\right) + \mu\left(\left\{ x \in X : |g(x)| > \frac{\lambda}{2}\right\}\right).
\]
Thus
\[
\lambda^p D_{f+g}(\lambda) \leq \lambda^p D_{f}\left(\frac{\lambda}{2}\right) + \lambda^p D_{g}\left(\frac{\lambda}{2}\right)
\leq 2^p \left[\sup_{\lambda > 0} \lambda^p D_f(\lambda) + \sup_{\lambda > 0} \lambda^p D_g(\lambda)\right] = 2^p \left[||f||_{L^p_w} + ||g||_{L^p_w}\right].
\]
Therefore
\[
||f + g||_{L^p_w} = \left(\sup_{\lambda > 0} \lambda^p D_{f+g}(\lambda)\right)^{\frac{1}{p}} \leq 2 \left[||f||_{L^p_w} + ||g||_{L^p_w}\right]^{\frac{1}{p}}.
\]
The previous proposition shows that $L^p_w$ is a vector space and that $\|\cdot\|_{L^p_w}$ is a quasi-norm on it (which is a functional that is like a norm except that it does only satisfy the triangle inequality with a constant $c \geq 1$, that is $\|f + g\| \leq c(\|f\| + \|g\|)$.)

To proceed to the main theorem, we need the following lemma.

**Lemma 2.2.** If $I \subset \mathbb{R}$ has finite Lebesgue measure, then $\chi_I$, the characteristic function of $I$, belongs to $L^p_w$ for all $0 < p < \infty$.

**Proof.** For all $\lambda > 0$ we have

$$m \left\{ x \in \mathbb{R} : |\chi_I(x)| > \lambda \right\} = \begin{cases} 0, & \text{if } \lambda > 1 \\ m(I), & \text{if } 0 < \lambda \leq 1 \end{cases}.$$ 

Thus

$$\|\chi_I\|_{L^p_w} = \left( \sup_{\lambda > 0} \lambda^p D_{\chi_I}(\lambda) \right)^{\frac{1}{p}} \leq (m(I))^{\frac{1}{p}}.$$ 

So $\chi_I \in L^p_w$.

**Theorem 2.3.** $L^p_w[0, 1]\setminus L^p[0, 1]$ is lineable in $L^p_w[0, 1]$, for all $0 < p < \infty$.

**Proof.** Let $0 < p < \infty$ be fixed. Define $f : [0, 1] \to \mathbb{R}$ as

$$f(x) = \begin{cases} x^{-\frac{1}{p}}, & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0 \end{cases}.$$ 

For each $n \in \mathbb{N}$ let $g_n = \chi_{[0, 1/n]}$. Let $n \in \mathbb{N}$ be fixed. By the argument after Proposition 2.2, $f \cdot g_n \in L^p_w[0, 1]$ for all $n \in \mathbb{N}$. On the other hand by remark 2.3, $f \cdot g_n \notin L^p[0, 1]$. Since for each $m, n \in \mathbb{N}$ with $m < n$, $m([0, 1/n] \setminus [0, 1/m]) \neq 0$, so for each $m, n \in \mathbb{N}$ with $m < n$, $f \cdot g_n \neq f \cdot g_m$. Therefore $\{f \cdot g_n, n \in \mathbb{N}\}$ is an infinite set. Again by the argument after Proposition 2.2 and above reasoning, span$\{f \cdot g_n, n \in \mathbb{N}\} \subseteq L^p_w[0, 1]\setminus L^p[0, 1] \cup \{0\}$. So $L^p_w[0, 1]\setminus L^p[0, 1]$ is lineable.

**Remark 2.4.** This trend of research seems to be very broad. One can ask about the dimension of functional linear space that is contained in $L^p_w[0, 1]\setminus L^p[0, 1]$. Another question that can be considered is that whether there exists an infinitely generated algebra of functions contained in $L^p_w[0, 1]\setminus L^p[0, 1]$ or not. Also changing the space $[0, 1]$ by a general measure space would contribute to new results. In general this paper is just a beginning.

**References**