Distributional Laplace Transform of $t^k f(t^r)$

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Abstract

In this note distributional Laplace transform of the function $t^k f(t^r)$ is obtained and the relation between the distributional one-sided Laplace transform of $t^k f(t^r)$, $r > 0$, and the distributional Hankel transform of $f(t^r)$ is established.

Keywords: Hankel Transform, Laplace Transform.

1 INTRODUCTION

The Integral equations

$$L\{f, s\} = \int_0^\infty e^{-st} f(t) dt, \text{ Re } s > 0 \quad (1.1)$$

and,

$$H_\mu \{f, y\} = \int_0^\infty t f(t) J_\mu(yt) dt, \quad y > 0 \quad (1.2)$$

are designed as the Laplace transform and the Hankel transform of order $\mu$ [5] of a function $f(t) \in L(0, \infty)$ respectively. Bhonsle [1] has established the following classical relation,

$$L\{t^k f(t), s\} = \int_0^\infty y H_\mu(f, y) L\{t^k J_\mu(yt), s\} dy \quad (1.3)$$

between the Laplace and Hankel transform for $k > -1$ and Re $s > 0$.

Bhonsle and Sonwane [4] have extended this result (1.3) to some class of distributions. Bhonsle [2] has extended his classical result (1.3) to another classical result.

$$L\{t^k f(t^r), s\} = \int_0^\infty y H_\mu(f, y) L\{t^k J_\mu(t^r y), s\} dy \quad (1.4)$$

For $k > -1$, Re $s > 0$ and $r > 0$. It is clear that for $r=1$ equation (1.4) reduces to (1.3).

We use the notation and terminology given in [6].

In this note an attempt has been made to extend the result (1.4) for some class of distributions and for $r=1$ it reduces to [4].
Lemma 1.1 The mapping $e^{-st} \rightarrow t^k e^{-st}$ is a linear and continuous from $\mathscr{E}$ into itself, for $\text{Re } s > 0$ and $k = 0, 1, 2, 3, \ldots$.

Proof: First of all we shall show that $e^{-st} \in \mathscr{E}$ for $\text{Re } s > 0$. For each compact subset $K$ of $\mathbb{R}^n$ and for each non negative integer $k \in \mathbb{R}^n$,

We have,

$$\Gamma_{K,k}(e^{-st}) = \sup_{t \in K} |D^k e^{-st}|$$

$$= |s^k| \Gamma_{K,0}(e^{-st}) < \infty, \text{ for } \text{Re } s > 0.$$ 

Therefore $e^{-st} \in \mathscr{E}$.

Define

$$m(t) = \begin{cases} 0, & -\infty < t \leq 0 \\ t^k, & 0 < t < \infty. \end{cases}$$

Then $m(t)$ is smooth on $-\infty < t < \infty$ and $m(t) \in \theta_M$, the space of multipliers [6], since for each non negative integer $k$ there exists $N_k > 0$ such that $(1+t^2)^{-N_k} D^k m(t) < \infty$.

Now to show $e^{-st} \rightarrow t^k e^{-st}$ is linear and continuous, it is sufficient to show that $t^k e^{-st} \in \mathscr{E}$.

Consider

$$\Gamma_{K,k}(t^k e^{-st}) = \sup_{t \in K} |D^k (t^k e^{-st})|$$

$$= \sup_{t \in K} | \sum_{n=0}^{k} (n^k) \frac{K!}{(k-n)!} t^{k-n}(-1)^n s^n e^{-st} |$$

$$\leq \sum_{n=0}^{k} (n^k) \frac{k!|s^n|}{(k-n)!} \sup_{t \in K} |t^{k-n} e^{-st}|$$

$$= C_{k,s} \Gamma_{k,0}(t^{k-n} e^{-st})$$

where

$$C_{k,s} = \sum_{n=0}^{k} (n^k) \frac{k!|s^n|}{(k-n)!}$$

Linearity of the mapping $e^{-st} \rightarrow t^k e^{-st}$ is obvious and its continuity is implied by [6].

Theorem 1.1 If $f \in \mathscr{E}'$ and $\mu \geq -\frac{1}{2}$, then
where,

\[ F(y) = \langle f(t^r), t J(t^r y) \rangle, \quad r > 0 \]

and,

\[ \Phi(s, y) = yL_+ \{ t^k J_\mu(t^r y) \} \]

**Proof:** From above Lemma if \( e^{-st} \to t^k e^{-st} \) is linear and continuous from \( \mathcal{E} \) into itself then the adjoint mapping \( f(t^r) \to t^k f(t^r) \) is a linear and continuous mapping [6] from \( \mathcal{E}' \) into itself and is defined by,

\[ \langle t^k f(t^r), e^{-st} \rangle = \langle f(t^r), t^k e^{-st} \rangle \]

where \( f(t^r) \in \mathcal{E}' \), \( e^{-st} \in \mathcal{E} \) and \( t^k \) is multiplier for \( \mathcal{E}' \).

Note that when \( f \in \mathcal{E}' \) and \( \mu \geq \frac{1}{2} \), \( F(y) = \langle f(t^r), t J_\mu(t^r y) \rangle \) is a smooth function on \( 0 < y < \infty \).

Then

\[ f(t^r) = \lim_{Y \to \infty} \int_0^Y F(y) y J_\mu(t^r y) dy \]

in the sense of convergence in \( \mathcal{E}'(I) \). Hence

\[ \langle t^k f(t^r), e^{-st} \rangle = \langle \lim_{Y \to \infty} \int_0^Y F(y) y J_\mu(t^r y) dy, t^k e^{-st} \rangle \]

Since \( y J_\mu(t^r y) \) is bounded on \( 0 < t^r y < \infty \), \( t^k e^{-st} \in \mathcal{E} \)

and \( F(y) \) is smooth on \( 0 < y < \infty \), we can write

\[ \langle t^k f(t^r), e^{-st} \rangle = \langle F(y), \lim_{Y \to \infty} \int_0^\infty y J_\mu(t^r y) t^k e^{-st} dt \rangle \]

\[ = \langle F(y), yL_+ [t^k J_\mu(t^r y)] \rangle \]

Therefore,

\[ \langle t^k f(t^r), e^{-st} \rangle = \langle F(y), \Phi(s, y) \rangle \]

where,

\[ \Phi(s, y) = yL_+ [t^k J_\mu(t^r y)] \]
References


