SOLITON SOLUTIONS OF A GENERALIZED HIROTA-SATSUMA EQUATION USING DARBOUX TRANSFORMATION

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Abstract: A modified version of the generalized Hirota-Satsuma equation is solved analytically using Darboux transformations (DT). We start with the Lax pair of this equation and apply DT. This leads to another solvable pair containing new eigenfunctions that is a solution of the equation. Several seeds solutions are tested as well as one and two solitons forms are obtained using DT. A suitable choice of the seed fields leads to new solutions.

Keywords: Darboux transformation, Hirota-Satsuma equation, lax pair, solitons.

1. INTRODUCTION

The wave observed in plasma, elastic media, optical fibers, fluid dynamics are described by nonlinear partial differential equations. In the past decades, several methods for obtaining analytic solutions of nonlinear partial differential equations (NPDEs) have been presented, such as the inverse scattering method[1], Hirota's method [2], the Backlund and Darboux transformation [3,4], Painlevé expansions [5], homogeneous balance method [6], Jacobi elliptic function [7, 8], extended tanh-function methods[9-11], extended F-expansion methods [12-15], Adomain methods [16,17], Exp-function methods [18, 19] and finally the Mapping method [20-24].

In this paper, we solve using Darboux transformation method (25-30), the generalized Hirota–Satsuma equation in three dimensions (3D) described as follows;

\[
\left[h_{xxx} - \frac{3}{4} \left( \frac{\partial^2 h}{\partial x^2} \right)^2 + 3h_x h_{xx} \right]_x = h_{yz} \tag{1}
\]

This equation describes the flow of an incompressible fluid. Using the Singular Manifold Method (SMM), Estevez et al[31] derive its Lax pair in the form of;

\[-\psi_y + \psi_{xxx} + 3h \psi_x + \frac{3}{2} h_{xx} \psi = 0 \tag{2}
\]

\[2h \psi_{xzz} - h_x \psi_z + 2h_z \psi_y = 0 \tag{3}
\]

where \(h(x, y, z)\) is the wave amplitude and \(\psi(x, y, z)\) in the system (2) and (3) is eigenfunctions. The present work is organized as follows; section2, is devoted to the mathematical formulation of the problem. We start with an initial solution \(h\) and recursively obtain via the system eigenfunctions \(\psi, \psi_1\), an improved solution \(h[1]\). Applying DT, \(N\)-times, produces \(N\)-soliton solutions. In Section 3, we explicit detail, the explicit solitary wave solutions for different seeds form and plot them. In Section 4, two soliton solutions are derived and plotted, applying the DT method. The paper ends with a conclusion, in section (5).

2. MATHEMATICAL FORMULATION

Darboux transformation is a recursive algorithm, deriving a series of explicit solutions from a trivial one. Applying it to the Laxpair (2) and (3) results in two eigenfunction \(\psi, \psi_1\). These are used together with an initial seed solution \(h\) in the following equations;

\[
\psi[1] = \left( \frac{d}{dx} - \frac{\psi_1}{\psi} \right) \psi \tag{4}
\]

\[h[1] = h + \frac{\psi_1}{\psi} \tag{5}
\]

where \(\psi[1]\) satisfies (2) and (3) and \(h[1]\) is a new solution for equation (1). Replacing for \(\psi[1]\) in (2) and (3) yields;

\[-\psi_y + \psi_{xxx}[1] + 3h_x[1] \psi_x[1] + \frac{3}{2} h_{xx}[1] \psi[1] = 0 \tag{6}
\]

\[2h_x[1] \psi_{xzz} - h_{xzz}[1] \psi_x[1] + 2h_z[1] \psi_y[1] = 0 \tag{7}
\]

Applying DT, once more we have;

\[\psi[2] = \left( \frac{d}{dx} - \frac{\psi[1]}{\psi} \right) \psi[1] \tag{8}
\]

\[h[2] = h[1] + \frac{d}{dx} \ln \psi[1] \tag{9}
\]

where \(\psi[1]\) is defined as;

\[\psi[2] = \left( \frac{d}{dx} - \frac{\psi[1]}{\psi} \right) \psi[1] \tag{10}
\]
where $\psi_2$ is additional solution of (2) and (3) using a different constant of integration. From (8) into equations (6) and (7) we obtain:

$$-\psi_y[2] + \psi_{xxx}[2] + 3h_x[2]\psi_x[2] + \frac{3}{2}h_{xx}[2]\psi[2] = 0 \quad (11)$$

$$2h_x[2]\psi_{xx}[2] - h_{xx}[2]\psi_x[2] + 2h_2[2]\psi[2] = 0 \quad (12)$$

Where $h[2]$ is a new solution of (1). Applying DT N-time gives the following forms for $\psi[N], h[N]$:

$$\psi[N] = \frac{W(\psi_1, \psi_2, ..., \psi_N, \psi)}{W(\psi_1, \psi_2, ..., \psi_N)} \quad (13)$$

$$h[N] = h + \frac{2}{\sigma_2} \ln W(\psi_1, \psi_2, ..., \psi_N) \quad (14)$$

where $W$ is the Wronskian of the eigenfunctions; $\psi_1, \psi_2, ..., \psi_N, \psi$.

3. SOLITARY WAVE SOLUTION

This section gives a singlesoliton (solitary wave) solution for both the nonlinear equation (1) and its Lax pair (2) and (3). To simplify the solution of this system, we use a simple seed field (h). Some seed fields are chosen and the explicit solutions are given below.

3.1 First initial (seed) solution

Consider an initial wave form:

$$h = x + y + z \quad (15)$$

Substituting $h$ into equations (2) and (3) gives:

$$-\psi_y + \psi_{xxx} + 3\psi_x = 0 \quad (16)$$

$$\psi_{xx} + \psi = 0 \quad (17)$$

Let in equation (16) the solitary wave solution be:

$$\psi(x, y, z) = \phi(\zeta). \quad (18)$$

Where, $\zeta = x + c_1y + \frac{x}{c_1} - \frac{3}{c_1}$ to reduce equation (16) as follow;

$$\phi(\zeta) + (3 - c_1)\phi_x = 0 \quad (19)$$

Integrating with respect to $\zeta$ we obtain:

$$\phi(\zeta) + (3 - c_1)\phi = c \quad (20)$$

Where, $c$ is an arbitrary constant of integration. As the boundary conditions for solitary wave are: $\phi, \phi_\zeta, \phi_{\zeta\zeta} \rightarrow 0$ as $\zeta \rightarrow \pm \infty$, thus $c=0$. Hence the solution of equation (20) will be:

$$\phi(\zeta) = A(z)e^{\lambda_1\zeta} + B(z)e^{-\lambda_1\zeta} \quad (21)$$

Where,

$$\lambda_1 = \sqrt{c_1 - 3}, \quad c_1 > 3 \quad (22)$$

and $A(z), B(z)$ are function to be determined later. Returning to the original variables $x, y$ and $\psi$, we have:

$$\psi(x, y, z) = k_1e^{\lambda_1x + c_1y - \frac{1}{\lambda_1}z} + k_2e^{-\lambda_1x + c_1y - \frac{1}{\lambda_1}z} \quad (23)$$

Where, $k_1, k_2$ are two arbitrary constants.

Let $k_1 = \frac{1}{2}, \quad k_2 = \frac{1}{2}$ we obtain:

$$\psi(x, y, z) = \sinh \left(\lambda_1(x + c_1y) - \frac{1}{\lambda_1}z\right) \quad (24)$$

Others values are $k_1 = \frac{1}{2}, \quad k_2 = \frac{1}{2}$ gives:

$$\psi(x, y, z) = \cosh \left(\lambda_1(x + c_1y) - \frac{1}{\lambda_1}z\right) \quad (25)$$

Hence from formula (4), $\psi[1]$ will be:

$$\psi[1] = sech \left(\lambda_1(x + c_1y) - \frac{1}{\lambda_1}z\right) \quad (26)$$

Which is a solution of the new pair (6) and (7) with new field $h[1]$ from (5) given by

$$h[1] = x + y + z + \lambda_1 \tanh \left(\lambda_1(x + c_1y) - \frac{1}{\lambda_1}z\right) \quad (27)$$

$h[1]$, the solitary solution of equation (1) is plotted below in Fig.1.
3.2 Second Seed Solution

Consider an initial wave form
\[ h = xy + \left( \frac{1}{2} \right) \quad (28) \]

Substituting \( h \) from (28) into system (2, 3) reduces it to the form
\[ -\psi_y + \psi_{xxx} + 3y\psi_x = 0 \quad (29) \]
\[ -x^2\psi_{xx} + \psi = 0 \quad (30) \]
Solving the system (29) and (30) yields;

\[ \psi_1 = \cosh \left( \lambda_1 x + \left( \frac{\lambda_1^2}{2} + \frac{3}{2} y \right) \lambda_1 y - \frac{1}{\lambda_1^2} \right) \quad (31) \]

From (31) in (7) we obtain the solution;

\[ h[1] = xy + \left( \frac{1}{2} \right) + \lambda_1 \ tanh \left[ \lambda_1 x + \lambda_1^2 y + \frac{3}{2} \lambda_1 y^2 - \frac{1}{\lambda_1^2} \right] \quad (32) \]

This solution is plotted below in Fig. 2 for \( \lambda_1 = 3, z = 2 \)
4. TWO-SOLITON SOLUTION
To derive a two-soliton solution, $h[2]$, we apply the DT formula for two soliton solution, formula (8) and (9) can be written as:

$$\psi[2] = \frac{W(\psi_1, \psi_2, \psi)}{W(\psi_1, \psi_2)}$$

$$h[2] = h + \frac{d}{dx} \ln W(\psi_1, \psi_2)$$

where $W$ is the wronskian of three eigenfunctions; $\psi_1, \psi_2, \psi_2$, while $h$ is the seed solution.

4.1 First seed solution
The first seed solution has the form; $h = x + y + z$, while $\psi_1, \psi_2$ are obtained from equations (24) and (25). To obtain $\psi_2$, we set $k_1 = k_2 = \frac{1}{2}$ in (23), and choose another $\lambda_2$ parameter as $\lambda_2 = \sqrt{c_2 - 3}, c_2 > 3$. This result in:

$$\psi_2(x, y, z) = \cosh \left( \frac{\lambda_2(x + c_2 y) - \frac{1}{\lambda_2} z}{\lambda_2} \right)$$

Replacing for $\psi_2$ in (33),(34) we obtain:

$$\psi[2] = \frac{\lambda_1(x^2 - \lambda_2^2)}{\lambda_2 \tanh\left( \frac{\lambda_2(x+c_2 y)-\frac{z}{\lambda_2}}{\lambda_2} \right) - \lambda_1 \tanh\left( \frac{\lambda_1(x+c_1 y)-\frac{z}{\lambda_1}}{\lambda_1} \right)}$$

$$h[2] = x + c_1 y + z + \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2 \tanh\left( \frac{\lambda_2(x+c_2 y)-\frac{z}{\lambda_2}}{\lambda_2} \right) - \lambda_1 \tanh\left( \frac{\lambda_1(x+c_1 y)-\frac{z}{\lambda_1}}{\lambda_1} \right)}$$

The two-soliton solution, $h[2]$ is plotted in Fig.3 for different $c$’s.
4.2 SECOND SEED SOLUTION $h$

Starting with another initial wave form;

$$h = xy + \frac{1}{x} \tag{38}$$

And following steps similar to those in § 4.1, we obtain;

$$h[2] = xy + \frac{1}{x} + \lambda_1 \tanh \left( \lambda_1 x + \frac{\lambda^3}{2} y^2 - \frac{1}{\lambda_1 z} \right) + \frac{\lambda_2 - \lambda_1^2 + \lambda_2^2 \tanh^2 \left( \lambda_2 x + \lambda_3 \frac{3}{2} y^2 - \frac{1}{\lambda_2 z} \right)}{\lambda_2 \tanh \left( \lambda_2 x + \lambda_3 \frac{3}{2} y^2 - \frac{1}{\lambda_2 z} \right) - \lambda_1 \tanh \left( \lambda_1 x + \lambda_3 \frac{3}{2} y^2 - \frac{1}{\lambda_1 z} \right) - \lambda_2 \tanh \left( \lambda_2 x + \lambda_3 \frac{3}{2} y^2 - \frac{1}{\lambda_2 z} \right)}$$

This solution $h[2]$ is plotted in Fig. 4 for $\lambda_1 = 0.5, \lambda_2 = 0.25, z = 1$.
5. CONCLUSIONS
We did here solve HirotaSatsuma equation (1) using Darboux transformation. We tested different seed solutions and applying one and two solitons DT, we derive new solutions.

REFERENCES