FIXED POINT THEOREMS FOR RATIONAL TYPE CONTRACTIONS WITH PPF DEPENDENCE IN BANACH SPACES

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Abstract: The aim of this paper is to prove fixed point theorem with PPF dependence for mappings involving \((\varphi, \phi)\) - rational type contraction in Razumikhin class.

Keywords: PPF dependent fixed point, rational type contraction, Razumikhin class.

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1. Introduction: Fixed point theory is one of the well known traditional theories in mathematics that has a broad set of applications. In 1922, Polish mathematician Stephan Banach published his famous contraction principle. Since then, this principle has been extended and generalized in several ways either by using the contractive condition or by imposing some additional conditions on an ambient space. This principle is one of the cornerstones in the development of fixed point theory. From inspiration of this work, several mathematicians heavily studied this field. For example, the work of Kannan [11], Chatterjea [3], Borinde [1], Ciric [4], Geraghty [8], Meir and Keeler [13], Suzulci [15] and so forth. Das and Gupta [5] and Jaggi [9] were the pioneers in proving fixed point theorems using contractive conditions involving rational expressions.

On the other hand, Bernfeld et al. [2] introduced the concept of Past-Present-Future (for short PPF) dependent fixed point or the fixed point with PPF dependence which is one type of fixed points for mappings that have different domains and ranges. They also established the existence of PPF dependent fixed point theorems in the Razumikhin class for Banach type contraction mappings. These results are useful for proving the solutions of nonlinear functional differential and integral equations which may depend upon the past history, present data, and future consideration. Some papers about fixed point theorems with PPF dependence have appeared in the literature (see e.g., [6,7,10,12,14]).

In [14] authors present some fixed point theorems for contraction of rational type with PPF dependence. While in [12] authors introduced the new type of contraction mappings called Ciric-rational type contraction and gave sufficient condition for the existence of PPF dependent fixed point theorems in Razumikhin class. They apply their results to study the existence and uniqueness of solution of a nonlinear integral equation.

The aim of this paper, is to prove some fixed point theorems with PPF dependence by using a class of pairs of functions satisfying certain assumptions.
2. PRELIMINARIES: In this section, we recall some concepts and definitions that will be required in the sequel. Throughout this paper, $E$ will denote a Banach space with the norm $\| \cdot \|_E$, and $E_0 = C([a,b], E)$ will denote the space of the continuous $E$-valued functions defined on $[a,b]$ and equipped with the norm $\| \cdot \|_{E_0}$ given by

$$\| \phi \|_{E_0} = \sup_{t \in [a,b]} \| \phi(t) \|_E : \text{ for } \phi \in E_0.$$  

Let $T : E_0 \to E$ be a mapping. A point $\phi \in E_0$ is said to be a PPF dependence fixed point of $T$ or a fixed point with PPF dependence of $T$ if $T\phi = \phi(c)$ for some $c \in [a,b]$.

For a fixed element $c \in [a,b]$, the Razumikhin class $\mathcal{R}_c$ is defined by

$$\mathcal{R}_c = \{ \phi \in E_0 : \| \phi \|_{E_0} = \| \phi(c) \|_E \}$$

Remark 2.1: Note that, for $x \in E$ fixed, the function $\phi_x$ defined by

$$\phi_x(t) = x \text{ for } t \in [a,b]$$

satisfies $\phi_x \in E_0$, $\| \phi_x \|_{E_0} = \| \phi_x(c) \|_E = \| x \|_E$ for any $c \in [a,b]$, and therefore $\phi_x \in \mathcal{R}_c$ for any $c \in [a,b]$. Consequently $\mathcal{R}_c \neq \phi$ for any $c \in [a,b]$.

We say that the class $\mathcal{R}_c$ is topologically closed with respect to difference if for any $\phi, \xi \in \mathcal{R}_c$, we have $\phi - \xi \in \mathcal{R}_c$.

Similarly, we say that the class $\mathcal{R}_c$ is topologically closed with respect to the topology on $E_0$ induced by the norm $\| \cdot \|_{E_0}$.

The Razumikhin class plays an important role in the existence of PPF dependent fixed point.

Definition 2.1 (see Bernfeld et al. [2]). The mapping $T : E_0 \to E$ is said to be Banach type contraction if there exists a real number $\alpha \in [0,1)$ such that

$$\| T\phi - T\xi \|_E \leq \alpha \| \phi - \xi \|_{E_0}$$

for all $\phi, \xi \in E_0$.

The following PPF dependent fixed point theorem is proved by Bernfeld et al. [2].

Theorem 2.1. ([2]) Let $T : E_0 \to E$ be a Banach type contraction. If $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference, then $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$.

3. MAIN RESULTS:

We start this section presenting the following class of pairs of functions $F$. A pair of functions $(\varphi, \phi)$ is said to belong to the class $F$ if they satisfy the following conditions:

(i) $\varphi, \phi : [0,\infty) \to [0,\infty)$;

(ii) For $t, s \in [0,\infty)$, if $\varphi(t) \leq \phi(s)$ then $t \leq s$;

(iii) For $(t_n)$ and $(s_n)$ sequence in $[0,\infty)$ such that $\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n = a$.

If $\varphi(t_n) \leq \phi(s_n)$ for any $n \in N$, then $a = 0$.

Remark 3.1. Notice that if $(\varphi, \phi) \in F$ and $\varphi(t) \leq \phi(t)$, then $t = 0$, since we can take $t_n = s_n = t$ for any $n \in N$ and by (iii) we deduce $t = 0$. 

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In the sequel, we present some interesting examples of pairs of functions belonging to the class $F$ which will be very important in our study.

**Example 3.1.** Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a continuous and increasing function such that $\varphi(t) = 0$ if and only if $t = 0$ (these functions are known in the literature as altering distance functions).

Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$ and suppose that $\phi \leq \varphi$. Then the pair $(\varphi, \varphi - \phi) \in F$. In fact, it is clear that $(\varphi, \varphi - \phi)$ satisfy (i).

To prove (ii), suppose that $t, s \in [0, \infty)$, and $\varphi(t) \leq (\varphi - \phi)(s)$. Then, from

$$\varphi(t) \leq \varphi(s) - \varphi(s) \leq \varphi(s)$$

and taking into account the increasing character of $\varphi$, we can deduce that $t \leq s$.

In order to prove (iii), we suppose that

$$\varphi(t_n) \leq \varphi(s_n) - \varphi(s_n) \leq \varphi(s_n)$$

for any $n \in N$.

Where $t_n, s_n \in [0, \infty)$, and

$$\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n = a$$

Taking $n \to \infty$ in (A), we infer that $\lim_{n \to \infty} \phi(s_n) = 0$.

Let us suppose that $a > 0$. Since $\lim_{n \to \infty} s_n = a > 0$, we can find $\varepsilon > 0$ and a subsequence $(s_{n_k})$ of $(s_n)$ such that $s_{n_k} > \varepsilon$ for any $k \in N$. As $\phi$ is non-decreasing, we have $\phi(s_{n_k}) > \phi(\varepsilon)$ for any $k \in N$ and, consequently, $\lim_{k \to \infty} \phi(s_{n_k}) \geq \phi(\varepsilon)$. This contradicts the fact that $\lim_{k \to \infty} \phi(s_{n_k}) = 0$. Therefore, $a = 0$.

This proves that $(\varphi, \varphi - \phi) \in F$.

An interesting particular case is when $\varphi$ is the identity mapping, $\varphi = 1_{[0, \infty)}$, and $\phi : [0, \infty) \rightarrow [0, \infty)$ is a non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) \leq t$ for any $t \in [0, \infty)$.

**Example 3.2.** Let $S$ be the class of functions defined by

$$S = \{ \alpha : [0, \infty) \rightarrow [0,1) : \alpha(t_n) \rightarrow 1 \Rightarrow t_n \rightarrow 0 \}.$$  

Let us consider the pairs of functions $(1_{[0, \infty)}, \alpha 1_{[0, \infty)})$, where $\alpha \in S$ and $\alpha 1_{[0, \infty)}$ is defined by $(\alpha 1_{[0, \infty)})(t) = \alpha(t)t$, for $t \in [0, \infty)$.

Then $(1_{[0, \infty)}, \alpha 1_{[0, \infty)}) \in F$ It is clear that the pairs $(1_{[0, \infty)}, \alpha 1_{[0, \infty)})$, with $\alpha \in S$ satisfy (i).

To prove (ii), from

$$1_{[0, \infty)}(t) \leq (\alpha 1_{[0, \infty)})(s) \quad \text{for} \quad t, s \in [0, \infty),$$

we infer, since $\alpha : [0, \infty) \rightarrow [0,1)$, that $t \leq \alpha(s)s < s$ and, consequently, $1_{[0, \infty)}, \alpha 1_{[0, \infty)})$, satisfies (ii).

In order to prove (iii), we suppose that

$$1_{[0, \infty)}(t_n) = t_n \leq (\alpha 1_{[0, \infty)})(s_n) = \alpha(s_n)s_n$$

for any $n \in N$.

Where $t_n, s_n \in [0, \infty)$, and $\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n = a$
Let us suppose that \( a > 0 \). Since \( \lim_{n \to \infty} s_{n_k} = a > 0 \), we can find a subsequence \( (s_{n_k}) \) such that \( s_{n_k} > 0 \) for any \( k \in \mathbb{N} \). Now, as 
\[
t_n \leq \alpha(s_n) s_n \leq s_n \quad \text{for any} \quad n \in \mathbb{N},
\]
in particular, we have 
\[
t_{n_k} \leq \alpha(s_{n_k}) s_{n_k} \leq s_{n_k} \quad \text{for any} \quad k \in \mathbb{N},
\]
and, since \( s_{n_k} > 0 \) for any \( k \in \mathbb{N} \),
\[
\frac{t_{n_k}}{s_{n_k}} \leq \alpha(s_{n_k}) \leq 1.
\]
Taking \( k \to \infty \) in the last inequality, we obtain 
\[
\lim_{k \to \infty} \alpha(s_{n_k}) = 1
\]
Finally, since \( \alpha \in \mathcal{S} \), we infer that \( \lim_{k \to \infty} (s_{n_k}) = 0 \) and this contradicts the fact that \( \lim_{n \to \infty} s_n = a > 0 \).
Therefore, \( a = 0 \).

This proves that \( \left\{ \alpha_{1(0, \infty)} \right\} \in F \) for \( \alpha \in \mathcal{S} \).

**Remark 3.2.** Suppose that \( g : [0, \infty) \to [0, \infty) \) is an increasing function and \( (\phi, \delta) \in F \). Then it is easily seen that the pair \( (g \circ \phi, g \circ \delta) \in F \).

Now, we are ready to present our main result.

**Theorem 3.1.** Let \( T : E_0 \to E \) be a mapping such that there exists a pair of functions \( (\phi, \delta) \in F \) and \( c \in [a, b] \) such that 
\[
\phi\left(\|T\xi - T\eta\|_E\right) \leq \max\left\{ \phi(\|\xi - \eta\|_{E_0}), \phi\left(\|\eta(c) - T\eta\|_E + \frac{1 + \|\xi(c) - T\xi\|_E}{1 + \|\xi - \eta\|_{E_0}}\right)\right\}
\]
for any \( \xi, \eta \in E_0 \).

If \( \mathcal{R}_c \) is topologically closed and algebraically closed with respect to difference, then \( T \) has a unique PPF dependent fixed point in \( \mathcal{R}_c \).

**Proof:** Let \( \xi_0 \) be an arbitrary function in \( \mathcal{R}_c \). Since \( \xi_0 \in \mathcal{R}_c \subset E_0 \) put \( x_1 = T\xi_0 \in E \). By Remark 2.1, we may choose \( \xi_1 \in \mathcal{R}_c \) such that 
\[
T\xi_0 = x_1 = \xi_1(c).
\]
(1)

Since \( \xi_1 \in \mathcal{R}_c \subset E_0 \), put \( x_2 = T\xi_1 \in E \).

Using the same argument, we can find \( \xi_2 \in \mathcal{R}_c \) such that 
\[
T\xi_1 = x_2 = \xi_2(c).
\]
(2)

Continuing in this way, we can obtain a sequence \( \left\{ \xi_n \right\} \) in \( \mathcal{R}_c \) such that 
\[
T\xi_{n-1} = \xi_n(c) \quad \text{for any} \quad n \in \mathbb{N}
\]
(3)

Since \( \mathcal{R}_c \) is algebraically closed with respect to difference, we have
\|
\bar{\xi}_p - \bar{\xi}_q\|_{E_0} = \|
\xi_p (c) - \xi_q (c)\|_{E} \text{ for any } p, q \in N. \tag{4}

First, we will prove that \(\lim_{n \to \infty} \|\bar{\xi}_{n+1} - \bar{\xi}_n\|_{E_0} = 0\).

From (3) and (4), we have
\[
\|\bar{\xi}_{n+1} - \bar{\xi}_n\|_{E_0} = \|\bar{\xi}_{n+1} (c) - \bar{\xi}_n (c)\|_{E} = \|T\bar{\xi}_n - T\bar{\xi}_{n-1}\|_{E} \text{ for any } n \in N, \tag{5}
\]

And therefore applying the contractive condition.

\[
\phi\left(\|\bar{\xi}_{n+1} - \bar{\xi}_n\|_{E_0}\right) = \phi\left(\|T\bar{\xi}_n - T\bar{\xi}_{n-1}\|_{E} \right) \leq \max\left\{ \phi\left(\|\bar{\xi}_n - \bar{\xi}_{n-1}\|_{E_0}\right), \phi\left(\|\bar{\xi}_{n-1} (c) - T\bar{\xi}_{n-1}\|_{E} \right) \right\} \frac{1 + \|\bar{\xi}_n (c) - T\bar{\xi}_n\|_{E}}{1 + \|\bar{\xi}_n - \bar{\xi}_{n-1}\|_{E_0}} \tag{6}
\]

Now, assume that there exists \(n_0 \in N\) such that \(\|\bar{\xi}_{n_0+1} - \bar{\xi}_{n_0}\|_{E_0} = 0\).

In this case, \(\bar{\xi}_{n_0+1} = \bar{\xi}_{n_0}\) and consequently \(\bar{\xi}_{n_0+1} (c) = \bar{\xi}_{n_0} (c)\).

Again from (3), we have
\[
T\bar{\xi}_{n_0} = \bar{\xi}_{n_0+1} (c) = \bar{\xi}_{n_0} (c), \tag{7}
\]

and \(\bar{\xi}_{n_0}\) would be the PPF dependent fixed point.

Now, assume that \(\|\bar{\xi}_{n+1} - \bar{\xi}_n\|_{E_0} \neq 0\) for any \(n \in N\).

Now there are two cases.

Case 1. Consider
\[
\max\left\{ \phi\left(\|\bar{\xi}_n - \bar{\xi}_{n-1}\|_{E_0}\right), \phi\left(\|\bar{\xi}_{n-1} (c) - T\bar{\xi}_{n-1}\|_{E} \right) \right\} \frac{1 + \|\bar{\xi}_n (c) - T\bar{\xi}_n\|_{E}}{1 + \|\bar{\xi}_n - \bar{\xi}_{n-1}\|_{E_0}} \tag{8}
\]

In this case from (6), we have
\[
\phi\left(\|\bar{\xi}_{n+1} - \bar{\xi}_n\|_{E_0}\right) \leq \phi\left(\|\bar{\xi}_n - \bar{\xi}_{n-1}\|_{E_0}\right) \tag{9}
\]

And since \((\phi, \phi) \in F\) we deduce that
\[
\|\bar{\xi}_{n+1} - \bar{\xi}_n\|_{E_0} \leq \|\bar{\xi}_n - \bar{\xi}_{n-1}\|_{E_0} \tag{10}
\]

Case 2. Consider
\[
\max\left\{ \phi\left(\|\bar{\xi}_n - \bar{\xi}_{n-1}\|_{E_0}\right), \phi\left(\|\bar{\xi}_{n-1} (c) - T\bar{\xi}_{n-1}\|_{E} \right) \right\} \frac{1 + \|\bar{\xi}_n (c) - T\bar{\xi}_n\|_{E}}{1 + \|\bar{\xi}_n - \bar{\xi}_{n-1}\|_{E_0}} \tag{11}
\]
\[ \phi \left( \left\| \xi_{n+1} - c \right\| - T \left\| \xi_n - c \right\| \right) \leq \frac{1 + \left\| \xi_n - c \right\| - T \left\| \xi_n - c \right\| \xi_n}{1 + \left\| \xi_n - \xi_{n-1} \right\| E_0} \tag{11} \]

In this case from (6) and \((\phi, \phi) \in F\), we have

\[
\left\| \xi_{n+1} - \xi_n \right\| E_0 \leq \left\| \xi_{n-1} - \xi_n \right\| E_0 + \frac{1 + \left\| \xi_n - \xi_{n+1} \right\| E_0}{1 + \left\| \xi_n - \xi_{n-1} \right\| E_0} \tag{12} \]

From (3) and (4), we get

\[
\left\| \xi_{n+1} - \xi_n \right\| E_0 \leq \left\| \xi_{n-1} - \xi_n \right\| E_0 + \frac{1 + \left\| \xi_n - \xi_{n+1} \right\| E_0}{1 + \left\| \xi_n - \xi_{n-1} \right\| E_0} \]

Since \( \left\| \xi_{n+1} - \xi_n \right\| E_0 \neq 0 \), from the last inequality, it follows that

\[
\left\| \xi_{n+1} - \xi_n \right\| E_0 \leq \left\| \xi_{n-1} - \xi_n \right\| E_0 \tag{13} \]

From both cases, we conclude that the sequence \( \left\{ \left\| \xi_{n+1} - \xi_n \right\| E_0 \right\} \) is a decreasing sequence of nonnegative real numbers.

Let \( r = \lim_{n \to \infty} \left\| \xi_{n+1} - \xi_n \right\| E_0 \), where \( r \geq 0 \) and denote

\[
A = \{ n \in N : n \text{satisfies (8)} \}
\]

\[
B = \{ n \in N : n \text{satisfies (11)} \} \tag{14} \]

We note that the following

1. If \( \text{card} A = \infty \), then from (6) we can find infinitely many natural numbers \( n \) satisfying inequality (9) and since
   \[
   \lim_{n \to \infty} \left\| \xi_{n+1} - \xi_n \right\| E_0 = \lim_{n \to \infty} \left\| \xi_n - \xi_{n-1} \right\| E_0 = r \]
   and \((\phi, \phi) \in F\), we have \( r = 0 \).

2. If \( \text{card} A = \infty \), then from (6) we can find infinitely many \( n \in N \) such that
   \[
   \phi \left( \left\| \xi_{n+1} - \xi_n \right\| E_0 \right) \leq \phi \left( \left\| \xi_{n-1} - c \right\| - T \left\| \xi_n - c \right\| \right) \left(1 + \left\| \xi_n - c \right\| - T \left\| \xi_n - c \right\| \right) \tag{15} \]

Since \((\phi, \phi) \in F\) and using the similar argument to the one used in case (2), we obtain

\[
\left\| \xi_{n+1} - \xi_n \right\| E_0 \leq \left\| \xi_{n-1} - \xi_n \right\| E_0 + \frac{1 + \left\| \xi_n - \xi_{n+1} \right\| E_0}{1 + \left\| \xi_n - \xi_{n-1} \right\| E_0} \tag{16} \]

For infinitely many \( n \in N \).

Letting \( n \to \infty \) in (16) and taking into account that

\[
\lim_{n \to \infty} \left\| \xi_{n+1} - \xi_n \right\| E_0 = r \]

we deduce that
\[ r \leq r \frac{1 + r}{1 + r} \]
And consequently, we obtain \( r = 0 \).
Therefore
\[
\lim_{n \to \infty} \| \xi_n - \xi_n^{E_0} \| = 0
\]
(17)
Next, we will show that \( \{ \xi_n \} \) is a Cauchy sequence in \( E_0 \).

In contrary case since \( \lim_{n \to \infty} \| \xi_n - \xi_n^{E_0} \| = 0 \), by Lemma 2.1 of [8], we can find \( \varepsilon > 0 \) and subsequences \( \{ \xi_{n(k)} \} \) and \( \{ \xi_{m(k)} \} \)
of \( \{ \xi_n \} \) satisfying

(i) \( n(k) > m(k) \geq k \) for \( k > 0 \);

(ii) \( \varepsilon \leq \| \xi_n - \xi_m \|_{E_0} \), \( \| \xi_{n(k)} - 1 - \xi_{m(k)} \|_{E_0} < \varepsilon \) for \( k > 0 \);

(ii) \( \lim_{k \to \infty} \| \xi_n - \xi_m \|_{E_0} = \lim_{k \to \infty} \| \xi_{n(k)} - \xi_{m(k)+1} \|_{E_0} \)
\[
= \lim_{k \to \infty} \| \xi_{n(k)} - \xi_{m(k)} \|_{E_0}
= \lim_{k \to \infty} \| \xi_{n(k)+1} - \xi_{m(k)+1} \|_{E_0} = \varepsilon
\]
(18)
Since \( \xi_{n(k)+1}, \xi_{m(k)+1} \in \mathcal{R}_E \) for any \( k \in N \). From (3) and (4), we have
\[
\| \xi_{n(k)+1} - \xi_{m(k)+1} \|_{E_0} = \| \xi_{n(k)+1} - \xi_{m(k)+1} \|_{E_0}
= \| T \xi_{n(k)} - T \xi_{m(k)} \|_{E_0} \quad \text{for any} \ k \in N .
\]
(19)
Now using contractive condition and from (3) and (4), we obtain
\[
\phi \| \xi_{n(k)+1} - \xi_{m(k)+1} \|_{E_0} = \phi \| T \xi_{n(k)} - T \xi_{m(k)} \|_{E_0}
\]
\[
\leq \max \left\{ \phi \left( \| \xi_{n(k)} - \xi_{m(k)} \|_{E_0} \right) \phi \left( \| \xi_{m(k)}(c) - T \xi_{m(k)} \|_{E} \frac{1 + \| \xi_{n(k)}(c) - T \xi_{m(k)} \|_{E}}{1 + \| \xi_{n(k)} - \xi_{m(k)} \|_{E_0}} \right) \right\}
\]
(20)
for any \( k \in N \).
Put \( C = \{ k \in N : \phi \left\| \xi_{m(k)} - \xi_{m+1(k)} \right\|_{E_0} \leq \phi \left\| \xi_{n(k)} - \xi_{m(k)} \right\|_{E_0} \} \).

\[
D = \left\{ k \in N : \phi \left\| \xi_{m(k)} - \xi_{m+1(k)} \right\|_{E_0} \leq \phi \left( \left\| \xi_{m(k)} - \xi_{m+1(k)} \right\|_{E_0} \cdot \frac{1}{1 + \left\| \xi_{n(k)} - \xi_{m(k)} \right\|_{E_0}} \right) \right\}
\]

By (20) we have \( \text{card} C = \infty \) or \( \text{card} D = \infty \).

Let us suppose that \( \text{card} C = \infty \). Then there exists infinitely many \( k \in N \) such that

\[
\phi \left\| \xi_{m(k)} - \xi_{m+1(k)} \right\|_{E_0} \leq \phi \left( \left\| \xi_{m(k)} - \xi_{m+1(k)} \right\|_{E_0} \cdot \frac{1}{1 + \left\| \xi_{n(k)} - \xi_{m(k)} \right\|_{E_0}} \right)
\]

(22)

And since \((\phi, \phi) \in F^*\), we have

\[
\lim_{k \to \infty} \left\| \xi_{m(k)} - \xi_{m+1(k)} \right\|_{E_0} = \lim_{k \to \infty} \left\| \xi_{m(k)} - \xi_{m+1(k)} \right\|_{E_0} = \varepsilon
\]

We infer from (18) that \( \varepsilon = 0 \). This is a contradiction.

Now, if \( \text{card} D = \infty \), then we can find infinitely many \( k \in N \) such that

\[
\phi \left\| \xi_{m(k)} - \xi_{m+1(k)} \right\|_{E_0} \leq \phi \left( \left\| \xi_{m(k)} - \xi_{m+1(k)} \right\|_{E_0} \cdot \frac{1}{1 + \left\| \xi_{n(k)} - \xi_{m(k)} \right\|_{E_0}} \right)
\]

(24)

And since \((\phi, \phi) \in F^*\), we obtain from the above inequality

\[
\left\| \xi_{m(k)} - \xi_{m+1(k)} \right\|_{E_0} \leq \left\| \xi_{m(k)} - \xi_{m+1(k)} \right\|_{E_0} \cdot \frac{1}{1 + \left\| \xi_{n(k)} - \xi_{m(k)} \right\|_{E_0}}
\]

(25)

Taking \( k \to \infty \) and in view of (17) and (18), we obtain \( \varepsilon \leq 0 \) which is contradiction.

Therefore, since in both the cases \( \text{card} C = \infty \) and \( \text{card} D = \infty \), we obtain a contradiction. This shows that \( \{ \xi_n \} \) is a Cauchy sequence in \( E_0 \).

Now since \( E_0 \) is a Banach space, we can find \( \xi^* \in E_0 \) such that

\[
\lim_{n \to \infty} \xi_n = \xi^*.
\]

As \( \xi_n \in \mathcal{R}_c \) and \( \mathcal{R}_c \) is topologically closed, we have \( \xi^* \in \mathcal{R}_c \).

Next, we will show that \( \xi^* \) is PPF dependent fixed point of \( T \).

By the contractive condition, we obtain
\[ \phi\left(\left\| T_{\xi} - T_{\xi_n}\right\|_E\right) \leq \max \left\{ \phi\left(\left\| \xi^* - \xi_n\right\|_{E_0}\right), \phi\left(\left\| \xi_n(c) - T_{\xi_n}\right\|_E \frac{1 + \left\| \xi^*(c) - T_{\xi}^{*}\right\|_E}{1 + \left\| \xi^* - \xi_n\right\|_{E_0}}\right) \right\} \]

for any \( n \in N \).

Now there are two cases again.

(1) There exist infinitely many \( n \in N \) such that

\[ \phi\left(\left\| T_{\xi} - T_{\xi_n}\right\|_E\right) \leq \phi\left(\left\| \xi^* - \xi_n\right\|_{E_0}\right) \]  

since \((\varphi, \phi) \in F\), in this case we obtain

\[ \phi\left(\left\| T_{\xi} - T_{\xi_n}\right\|_E\right) \leq \left\| \xi^* - \xi_n\right\|_{E_0} \]

For infinitely many \( n \in N \). Since \( \lim_{n \to \infty} \xi_n = \xi^* \), letting \( n \to \infty \) in the last inequality, we obtain

\[ \lim_{n \to \infty} T_{\xi_n} = T_{\xi}^{*} \]

Where, to simplify our consideration, we will denote the subsequence by the same symbol \( \{T_{\xi_n}\} \). By (3)

\[ T_{\xi}^{*} = \lim_{n \to \infty} T_{\xi_n} = \lim_{n \to \infty} \xi_{n+1}(c) \]

\[ \xi_n \to \xi^* \text{ in } E_0, \text{ this means that} \]

\[ \sup_{t \in [a,b]} \left\| \xi_n(t) - \xi^*(t)\right\|_E \to 0 \]

And consequently, \( \lim_{n \to \infty} \xi_{n+1}(c) = \xi^*(c) \).

From this last result and by (30), we deduce that

\[ T_{\xi}^{*} = \xi^*(c) \]

And therefore, \( \xi^* \) is PPF dependent fixed point of \( T \) in \( \mathcal{R}_\varepsilon \).

(2) There exist infinitely many \( n \in N \) such that

\[ \phi\left(\left\| T_{\xi} - T_{\xi_n}\right\|_E\right) \leq \phi\left(\left\| \xi_n(c) - T_{\xi_n}\right\|_E \frac{1 + \left\| \xi^*(c) - T_{\xi}^{*}\right\|_E}{1 + \left\| \xi^* - \xi_n\right\|_{E_0}}\right) \]

Again to simplify our consideration, we will denote the subsequence by the same symbol \( \{T_{\xi_n}\} \).

Since \((\varphi, \phi) \in F\), we deduce that

\[ \left\| T_{\xi} - T_{\xi_n}\right\|_E \leq \left\| \xi_n(c) - T_{\xi_n}\right\|_E \frac{1 + \left\| \xi^*(c) - T_{\xi}^{*}\right\|_E}{1 + \left\| \xi^* - \xi_n\right\|_{E_0}} \]

For any \( n \in N \), using (3), we have
\[ \left\| T\xi^* - \xi^* \right\|_E \leq \left\| \xi^*_n (c) - \xi^*_{n+1} (c) \right\|_E \cdot \frac{1 + \left\| \xi^*_n (c) - T\xi^* \right\|_E}{1 + \left\| \xi^* - \xi^* \right\|_E} \]  

(35)

For any \( n \in \mathbb{N} \), taking \( n \to \infty \) and by (17) we infer (29). From the above case, we deduce that \( \xi^* \) is PPF dependent fixed point of \( T \) in \( \Re_c \).

Therefore, in both cases we proved that \( \xi^* \) is PPF dependent fixed point of \( T \) in \( \Re_c \).

To prove uniqueness, suppose that \( \phi^* \) is another PPF dependent fixed point of \( T \) in \( \Re_c \). Then, since \( \phi^*, \xi^* \in \Re_c \) and \( \Re_c \) is algebraically closed with respect to difference, we obtain

\[ \left\| \xi^* - \phi^* \right\|_{E_0} = \left\| \xi^* (c) - \phi^* (c) \right\|_E \]  

(36)

As \( \xi^* (c) = T\xi^* \) and \( \phi^* (c) = T\phi^* \), we infer

\[ \left\| \xi^* - \phi^* \right\|_{E_0} = \left\| \xi^* (c) - \phi^* (c) \right\|_E = \left\| T\xi^* - T\phi^* \right\|_E \]  

(37)

Now using contractive condition, we have

\[ \phi \left( \left\| \xi^* - \phi^* \right\|_{E_0} \right) = \phi \left( \left\| T\xi^* - T\phi^* \right\|_E \right) \]

\[ \leq \max \left\{ \phi \left( \left\| \xi^* - \phi^* \right\|_{E_0} \right), \phi \left( \left\| \phi^* (c) - T\phi^* \right\|_E \cdot \frac{1 + \left\| \xi^* (c) - T\xi^* \right\|_E}{1 + \left\| \xi^* - \phi^* \right\|_{E_0}} \right) \right\} \]

\[ \leq \max \left\{ \phi \left( \left\| \xi^* - \phi^* \right\|_{E_0} \right), \phi (0) \right\} \]  

(38)

Again there are two cases.

(1) if \( \max \left\{ \phi \left( \left\| \xi^* - \phi^* \right\|_{E_0} \right), \phi (0) \right\} = \phi \left( \left\| \xi^* - \phi^* \right\|_{E_0} \right) \)

Then from (38) and since \( (\phi, \phi) \in F \), we obtain, using Remark 3.1

\[ \left\| \xi^* - \phi^* \right\| = 0 \]  

and therefore \( \xi^* = \phi^* \).

(2) if \( \max \left\{ \phi \left( \left\| \xi^* - \phi^* \right\|_{E_0} \right), \phi (0) \right\} = \phi (0) \)

Then from (38)

\[ \phi \left( \left\| \xi^* - \phi^* \right\|_{E_0} \right) \leq \phi (0) \]

Since \( (\phi, \phi) \in F \), we obtain \( \left\| \xi^* - \phi^* \right\|_{E_0} \leq 0 \).

Therefore \( \left\| \xi^* - \phi^* \right\|_{E_0} = 0 \) and consequently \( \xi^* = \phi^* \).

This completes the proof of the theorem.
By Theorem 3.1, we obtain the following corollaries.

**Corollary: 3.2.** Let \( T : E_0 \to E \) be a mapping such that there exists a pair of functions, since \((\varphi, \phi) \in F\) and \(c \in [a, b]\) satisfying

\[
\varphi(T_\xi - T_\eta_E) \leq \phi(\|\xi - \eta\|_{E_0})
\]

For any \( \xi, \eta \in E_0 \). If \( \mathcal{R}_c \) is topologically closed and algebraically closed with respect to difference, then \( T \) has a unique PPF dependent fixed point in \( \mathcal{R}_c \).

**Corollary: 3.3.** Let \( T : E_0 \to E \) be a mapping satisfying

\[
\|T_\xi - T_\eta\|_E = k \max \left\{ \|\eta(c) - T_\eta\|_E \left[ 1 + \|\xi(c) - T_\xi\|_E \right] \right\}
\]

For all \( \xi, \eta \in E_0 \).

If \( \mathcal{R}_c \) is topologically closed and algebraically closed with respect to difference, then \( T \) has a unique PPF dependent fixed point in \( \mathcal{R}_c \).

Taking into account Example 3.1, we have the following corollary.

**Corollary 3.4.** Let \( T : E_0 \to E \) be a mapping such that there exist two functions \( \varphi, \phi : (0, \infty) \to [0, \infty) \) and \( c \in [a, b] \) such that

\[
\varphi(T_\xi - T_\eta_E) = \max \left\{ \varphi(\|\xi - \eta\|_{E_0}) - \varphi(\|\xi - \eta\|_{E_0}) \varphi \left( \frac{\|\xi(c) - T_\xi\|_E \|\eta(c) - T_\eta\|_E}{1 + \|T_\xi - T_\eta\|_E} \right) \right\}
\]

For all \( \xi, \eta \in E_0 \) where \( \varphi \) is a continuous and increasing function satisfying \( \varphi(t) = 0 \) if and only if \( t = 0 \), and \( \phi \) is a non-decreasing function such that \( \phi(t) = 0 \) if and only if \( t = 0 \), and \( \phi \leq \varphi \).

If \( \mathcal{R}_c \) is topologically closed and algebraically closed with respect to difference, then \( T \) has a unique PPF dependent fixed point in \( \mathcal{R}_c \).

Corollary 3.4 has the following consequences.

**Corollary 3.5.** Let \( T : E_0 \to E \) be a mapping such that there exist two functions \( \varphi, \phi : [0, \infty) \to [0, \infty) \) and \( c \in [a, b] \) such that

\[
\varphi(T_\xi - T_\eta_E) \leq \varphi(\|\xi - \eta\|_{E_0}) - \varphi(\|\xi - \eta\|_{E_0})
\]
For all $\xi, \eta \in E_0$, where $\varphi$ is an increasing function and $\phi$ is a non-decreasing function and they satisfy $\varphi(t) = \phi(t) = 0$ if and only if $t = 0$, and $\varphi$ is continuous with $\phi \leq \varphi$.

If $\mathfrak{R}_c$ is topologically closed and algebraically closed with respect to difference, then $T$ has a unique PPF dependent fixed point in $\mathfrak{R}_c$.

**Corollary 3.5.** can be considered as the version, in the context of PPF dependent fixed point theorems, of the following result about fixed point theorems which appears in [16].

**Theorem 3.6.** (see [16]). Let $(X, d)$ be a complete metric space and $T : X \rightarrow X$ a mapping satisfying

$$\varphi(d(Tx, Ty)) \leq \varphi(d(x, y)) - \phi(d(x, y))$$

for $x, y \in X$, where $\varphi$ and $\phi$ satisfy the same conditions as in Corollary 3.5, Then $T$ has a unique fixed point.

**Corollary 3.7.** Let $T : E_0 \rightarrow E$ be a mapping such that there exist two functions $\varphi, \phi : [0, \infty) \rightarrow [0, \infty)$ satisfying the same condition as in corollary 3.5 and $c \in [a, b]$ such that

$$\varphi\left(\|T\xi - T\eta\|_E\right) = \varphi\left(\frac{\|\xi(c) - T\xi\|_E}{1 + \|T\xi - T\eta\|_E}\right) - \phi\left(\frac{\|\xi(c) - T\xi\|_E}{1 + \|T\xi - T\eta\|_E}\right).$$

For all $\xi, \eta \in E_0$. If $\mathfrak{R}_c$ is topologically closed and algebraically closed with respect to difference, then $T$ has a unique PPF dependent fixed point in $\mathfrak{R}_c$.

Taking into account Example 3.2, we have the following corollary.

**Corollary 3.7.** Let $T : E_0 \rightarrow E$ be a mapping such that there exist $\alpha \in S$ and $c \in [a, b]$ satisfying

$$\|T\xi - T\eta\|_E = \max\left\{\alpha\|\xi - \eta\|_{E_0}, \alpha\left(\frac{\|\xi(c) - T\xi\|_E}{1 + \|T\xi - T\eta\|_E}\right)\right\}.$$

For all $\xi, \eta \in E_0$. If $\mathfrak{R}_c$ is topologically closed and algebraically closed with respect to difference, then $T$ has a unique PPF dependent fixed point in $\mathfrak{R}_c$.

A consequence of Corollary 3.7 is the following result.

**Corollary 3.8.** Let $T : E_0 \rightarrow E$ be a mapping such that there exist $\alpha \in S$ satisfying

$$\|T\xi - T\eta\|_E \leq \alpha\|\xi - \eta\|_{E_0} \|\xi - \eta\|_{E_0}.$$

For all $\xi, \eta \in E_0$. If $c \in [a, b]$ such that $\mathfrak{R}_c$ is topologically closed and algebraically closed with respect to difference, then $T$ has a unique PPF dependent fixed point in $\mathfrak{R}_c$.

Corollary 3.8 is the version, in the context of PPF dependent fixed point theorems, of the following result about fixed point theorems appearing in [8].
Theorem 3.9. Let \((X, d)\) be a complete metric space and \(T : X \to X\) a mapping satisfying

\[
d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)
\]

for \(x, y \in X\), where \(\alpha \in S\). Then \(T\) has a unique fixed point.

REFERENCES:


