ON \( \text{gr}^*\)-HOMEO MORPHISM IN TOPOLOGICAL SPACES

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Abstract: This paper deals with \( \text{gr}^*\)-closed maps, \( \text{gr}^*\)-open maps, \( \text{gr}^*\)-homeomorphism, \( \text{gr}^{**}\)-homeomorphism and study their properties. Using these new types of maps, several characterizations and properties have been obtained.

Keywords: \( \text{gr}^*\)-closed maps, \( \text{gr}^*\)-open maps, \( \text{gr}^*\)-homeomorphism, and \( \text{gr}^{**}\)-homeomorphism.

1. INTRODUCTION

Generalized closed mappings were introduced and studied by Malghan[9]. Generalized open maps, rg-closed maps, g*-closed maps, g*-open maps, gpr-closed maps have been introduced and studied by sundaram [15], Arockiarani [1], shiek John[13], and Gnanamal[3] respectively. We give the definitions of some of them which are used our present study. The purpose of this paper is to introduce the concept of new class of maps called \( \text{gr}^*\)-closed maps and \( \text{gr}^*\)-open maps. Further we introduce \( \text{gr}^*\)-homeomorphism, \( \text{gr}^{**}\)-homeomorphism and discuss their properties.

2. PRELIMINARIES

Definition 2.1 A subset \((x, \tau)\) is said to be
1) g-closed [8] set if, \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).
2) Regular open [14] if \(A = \text{int}(\text{cl}(A))\).
3) \(\text{gr}^*\)-closed [4] if \(Rcl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is g-open in \(X\).
4) rg-closed [11] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \(X\).
5) gpr-closed [3] if \(pcl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular in \(X\).

The complements of the above mentioned closed sets and their respective open sets.

Definition 2.2 A map \(f: X \rightarrow Y\) is said to be
1) Continuous function [1] if \(f^{-1}(V)\) is closed in \(X\) for every closed set \(V\) in \(Y\).
2) g-Continuous function [2], if \(f^{-1}(V)\) is g-closed in \(X\) for every closed set \(V\) in \(Y\).
3) rg-continuous [11] if \(f^{-1}(V)\) is rg-closed in \(X\) for every closed set \(V\) in \(Y\).
4) gpr-continuous [3] if \(f^{-1}(V)\) is gpr-closed in \(X\) for every closed set \(V\) in \(Y\).
5) \(\text{gr}^*\)-continuous [5] if \(f^{-1}(V)\) is \(\text{gr}^*\)-closed in \(X\) for every closed set \(V\) in \(Y\).

Definition 2.3 A topological space \((X, \tau)\) is said to be
i) $T_{1/2}$ space if every closed set is closed.

ii) a $T_{gr}$ space if every $gr^*$-closed set is closed.

**Definition 2.4** A bijective function $f$: $(X,\tau)\rightarrow (Y,\sigma)$ is called

i) $g$-homeomorphism [10] if both $f$ and $f^\prime$ are $g$-continuous.

ii) $rg$-homeomorphism [11] if both $f$ and $f^\prime$ are $rg$-continuous.

iii) $gp^*$-homeomorphism [3] if both $f$ and $f^\prime$ are $gp^*$-continuous.

**Definition 2.5** A map $f:(X,\tau)\rightarrow (Y,\sigma)$ is called

(i) $R$-closed map (R-open map) [7], if the image $f(A)$ is $R$-closed (R-open) in $(Y,\sigma)$ for each closed (open) set $A$ in $(X,\tau)$.

(ii) $\pi g\theta$-closed [6], if the image of every closed set in $(X,\tau)$ is $\pi g\theta$-closed in $(Y,\sigma)$.

(iii) $gp^*$-closed map (briefly $gp^*$-closed) [12], if the image of every closed set in $X$ is $gp^*$-closed in $Y$.

(iv) Regular generalized $\alpha$-closed (briefly, $r g a$-closed) [16], if the image of every closed set in $(X,\tau)$ is $rg a$-closed in $(Y,\sigma)$.

### 3. gr* - CLOSED MAP

**Definition 3.1** Let $f$: $(X,\tau)\rightarrow (Y,\sigma)$ is said to be generalized regular star (briefly $gr^*$) Closed map if the image of every closed set in $(X,\tau)$ is $gr^*$-closed in $(Y,\sigma)$.

**Theorem 3.2**

(i) Every closed map is $gr^*$-closed map.

(ii) Every $r$-closed map is $gr^*$-closed map.

(iii) Every $gr^*$-closed map is $g$-closed map.

(iv) Every $gr^*$-closed map is $rg$-closed map.

(v) Every $gr^*$-closed map is $gpr$-closed map.

Proof: Follows from the definition

**Remark: 3.3** The converse of the above theorem need not be true as seen from the following examples.

**Example: 3.4** (i) Let $X = \{a,b,c\}$, $\tau = \{\Phi, \{b\}, X\}$, $\sigma = \{\Phi, \{b,c\}, Y\}$ Let $f$ be an identity map such that $f$: $X\rightarrow Y$ then $f$ is $gr^*$-closed but not a closed map.

(ii) Let $X = \{a,b,c\}$, $\tau = \{\Phi, \{a\}, \{a,b\}, X\}$, and $\sigma = \{\Phi, \{c\}, \{a,c\}, Y\}$. Then define $f$: $X\rightarrow Y$ be an identity map, then $f$ is $gr^*$-closed map but not $r$-closed map.

(iii) Let $X = \{a,b,c\}$, $\tau = \{\Phi, \{b\}, \{b,c\}, \{a,b\}, X\}$, and $\sigma = \{\Phi, \{a\}, \{a,b\}, Y\}$. Define a map $f$: $X\rightarrow Y$ by $f(a) = b$, $f(b) = a$, $f(c) = c$ then $f$ is $g$-closed but not $gr^*$-closed map.

(iv) Let $X = \{a,b,c\}$, $\tau = \{\Phi, \{a\}, \{b,c\}, X\}$, and $\sigma = \{\Phi, \{a\}, \{c\}, \{a,c\}, Y\}$. Then $f$: $X\rightarrow Y$ be an identity map. Then $f$ is $rg$-closed map but not $gr^*$-closed map.

(v) Let $X = \{a,b,c\}$, $\tau = \{\Phi, \{a\}, \{b\}, \{a,b\}, X\}$, and $\sigma = \{\Phi, \{c\}, Y\}$. Then $f$: $X\rightarrow Y$ be an identity then $f$ is $g pr$ – closed map but not $gr^*$-closed map.

**Theorem 3.5** A map $f$: $(X,\tau)\rightarrow (Y,\sigma)$ is $gr^*$-closed if and only if for each subset $S$ of $(Y,\sigma)$ and each open set $U$ containing $f^{-1}(s)$ there is an $gr^*$-open set $V$ of $(Y,\sigma)$ such that $S\subseteq V$ and $f^{-1}(V) \subseteq U$. 

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Proof: Suppose $f$ is $gr^*$-closed set of $(X, \tau)$. Let $S \subseteq Y$ and $U$ be an open set of $(X, \tau)$ such that $f^-(S) \subseteq U$. Now $X-U$ is closed set in $(X, \tau)$. Since $f$ is $gr^*$-closed, $(X-U)$ is an $gr^*$-closed set in $(Y, \sigma)$. Then $V \subseteq Y$ if $f(X-U)$ is $gr^*$-open set in $(Y, \sigma)$. $f^-(S) \subseteq U$ implies $S \subseteq V$ and $f^-(V) = X - f^-(f(X-U)) \subseteq X-(X-U) = U$, i.e., $f^-(V) \subseteq U$. Conversely, let $F$ be a closed set of $(X, \tau)$. Then $f^-(f(F)) \subseteq F$ is an open set in $(X, \tau)$. By hypothesis, there exists an $gr^*$-open set $V$ in $(Y, \sigma)$ such that $f^-(f(F)) \subseteq V$ and $f^-(V) \subseteq F$ and so $F \subseteq f^-(f(V))$. Hence $V \subseteq (f^-(f(V)) \subseteq V$ which implies $f(F) \subseteq V$. Since $V$ is $gr^*$-closed, $f(F)$ is $gr^*$-closed. That is $f(F)$ is $gr^*$-closed in $(Y, \sigma)$. Therefore $f$ is $gr^*$-closed map.

Theorem 3.6 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is closed map and $g: (Y, \sigma) \rightarrow (Z, \eta)$ is $gr^*$-closed map. Then the composition $g \circ f: (X, \tau) \rightarrow (Z, \eta)$ is $gr^*$-closed map.

Proof: Let $F$ be any closed set in $(X, \tau)$. Since $f$ is a closed map, $f(F)$ is closed set in $(Y, \sigma)$. Since $g$ is $gr^*$-closed map, $g(f(F)) = g \circ f(F)$ is $gr^*$-closed set in $(Z, \eta)$. Thus $g \circ f$ is $gr^*$-closed map.

Remark 3.7 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is $gr^*$-closed map and $g: (Y, \sigma) \rightarrow (Z, \eta)$ is closed map, then the composition need not be an $gr^*$-closed map as seen from the following example.

Example 3.8 Let $X = Y = Z = \{a,b,c,d\}$, $\tau = \{\emptyset, \{c\}, \{a,b\}, \{a,b,c\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, Y\}$, $\eta = \{\emptyset, \{b\}, \{d\}, \{b,c\}, \{b,c,d\}, Z\}$. Define $f: (X, \tau) \rightarrow (Y, \sigma)$, be an identity map and $g: (Y, \sigma) \rightarrow (Z, \eta)$ by $g(a) = d$, $g(b) = b$, $g(c) = c$, $g(d) = a$. Then $f$ is $gr^*$-closed map and $g$ is a closed map. But $g \circ f: (X, \tau) \rightarrow (Z, \eta)$ is not an $gr^*$-closed map.

Theorem 3.9 If $f: (X, \tau) \rightarrow (Y, \sigma)$ and $g: (Y, \sigma) \rightarrow (Z, \eta)$ be two $gr^*$-closed maps where $(Y, \sigma)$ is $T_{gr^*}$-space. Then the composition $g \circ f: (X, \tau) \rightarrow (Z, \eta)$ is $gr^*$-closed.

Proof: Let $A$ be a closed set of $(X, \tau)$. Since $f$ is $g$-closed, $f(A)$ is $g$-closed in $(Y, \sigma)$, by hypothesis $f(A)$ is closed. Since $g$ is $gr^*$-closed map, $g(f(A)) = g \circ f(A)$ is $gr^*$-closed in $(Z, \eta)$. Thus $g \circ f$ is $gr^*$-closed.

Theorem 3.10 Let $f: (X, \tau) \rightarrow (Y, \sigma)$, $g: (Y, \sigma) \rightarrow (Z, \eta)$ be two mappings such that their composition $g \circ f: (X, \tau) \rightarrow (Z, \eta)$ be $gr^*$-closed map. Then the following statements are true

(i) If $f$ is continuous and surjective, Then $g$ is $gr^*$-closed.
(ii) If $g$ is $gr^*$-irresolute and injective, then $f$ is $gr^*$-closed.
(iii) If $f$ is $g$-continuous, surjective and $(X, \tau)$ is a $T_{1/2}$ space then $g$ is $gr^*$-closed.

Proof: (i) Let $A$ be a closed set of $(Y, \sigma)$. Since $f$ is continuous, $f^{-1}(A)$ is closed in $(X, \tau)$. $g \circ f$ is $gr^*$-closed, therefore $g \circ f(f^{-1}(A))$ is $gr^*$-closed in $(Z, \eta)$. That is $g(A)$ is $gr^*$-closed in $(Z, \eta)$. Since $f$ is surjective. Therefore $g$ is $gr^*$-closed.

(ii) Let $A$ be a closed set of $(X, \tau)$. Since $g \circ f$ is $gr^*$-closed, $g \circ f(B)$ is $gr^*$-closed in $(Z, \eta)$. $g$ is $gr^*$- irresolute, $g^{-1}(g \circ f(B))$ is $gr^*$-closed set in $(Y, \sigma)$.

(iii) Let $c$ be a closed set of $(Y, \sigma)$. Since $f$ is $g$-continuous $f^{-1}(c)$ is closed in $(X, \tau)$. Since $(X, \tau)$ is a $T_{1/2}$ space, $f^{-1}(c)$ is closed. By hypothesis, $g(f^{-1}(c)) = g(c)$ is $gr^*$-closed in $(Z, \eta)$. Since $f$ is surjective. Therefore $g$ is $gr^*$-closed.

4. $gr^*$-OPEN MAP

Definition 4.1 A map $f: (X, \tau) \rightarrow (Y, \sigma)$ is called $gr^*$-open map if the image $f(A)$ is $gr^*$-open in $(Y, \sigma)$ for each open set $A$ in $(X, \tau)$.

Theorem 4.2 Every open map is $gr^*$-open but not conversely.

Proof: obvious.

Example 4.3 Let $X = Y = \{a,b,c\}$, $\tau = \{\emptyset, \{c\}, \{a,b\}, \{a,b,c\}, X\}$ $\sigma = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, Y\}$. Define $f: (X, \tau) \rightarrow (Y, \sigma)$ the identity map, then $f$ is $gr^*$-open but it is not an open map.

Theorem 4.4 For any bijection map $f: (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent.

a) $f^*: (Y, \sigma) \rightarrow (X, \tau)$ is $gr^*$-continuous.

b) $f$ is $gr^*$-open map and

c) $f$ is $gr^*$-closed map.

Proof: (a)â†’(b): Let $U$ be an open et of $(X, \tau)$. By assumption $(f^*)^{-1}(U) = f(U)$ is $gr^*$-open in $(Y, \sigma)$ and so $f$ is $gr^*$-open.

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(b)→(c): Let F be a closed set of (X, τ). Then \( F^C \) is open set on (X, τ). By hypothesis, \( f(F)^C \) is \( g^* \)-open in (Y, σ). That is, \( f(F)^C = f(F)^C \) is \( g^* \)-open in (Y, σ). Thus \( f(F) \) is \( g^* \)-closed in Y. Hence f is \( g^* \)-closed.

(c)→(a): Let F be a closed set in X. By hypothesis \( f(F) \) is \( g^* \)-closed in Y. That is \( f(F) = (f^{-1})^{-1}(F) \) and therefore \( f^{-1} \) is continuous.

5. \( g^* \)-homeomorphism

**Definition 5.1** A bijection \( f: (X, \tau) \to (Y, \sigma) \) is called generalized regular star (briefly, \( g^* \)) homeomorphism if \( f \) and \( f^{-1} \) are generalized regular star (briefly, \( g^* \)) continuous.

**Example 5.2** Consider \( X = Y = \{a, b, c, d\} \) with topology \( \tau = \{\emptyset, \{a, b\}, \{a, b, c\}, X\} \) and \( \sigma = \{\emptyset, \{b\}, \{a, c\}, \{a, b, c\}, Y\} \). Let \( f: X \to Y \) be a map defined by \( f(a) = a, f(b) = b, f(c) = c \) and \( f(d) = d \). Then \( f \) is bijective, \( g^* \)-continuous and \( f^{-1} \) is \( g^* \)-continuous. Hence \( f \) is \( g^* \)-homeomorphism.

**Theorem 5.3** Every \( g^* \)-homeomorphism is \( g^* \)-homeomorphism but not conversely.

**Proof:** Let \( f: (X, \tau) \to (Y, \sigma) \) be a homeomorphism. Then \( f \) and \( f^{-1} \) are continuous and \( f \) is bijection. Since every continuous function is \( g^* \)-continuous, \( f \) and \( f^{-1} \) are \( g^* \)-continuous. Hence \( f \) is \( g^* \)-homeomorphism.

**Example 5.4** Consider \( X = Y = \{a, b, c, d\} \) with topologies \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b, c\}, X\} \) and \( \sigma = \{\emptyset, \{b\}, \{b, c\}, \{a, b, c\}, Y\} \). Let \( f: X \to Y \) be the identity map. Then \( f \) is \( g^* \)-homeomorphism. But it is not a regular homeomorphism. Since the inverse image of the closed set \( \{c, d\} \) in \( X \) is \( \{c, d\} \) is not closed in \( Y \).

**Theorem 5.5** Every regular homeomorphism is \( g^* \)-homeomorphism, but not conversely.

**Proof:** The proof follows from the theorem 3.2.

**Example 5.6** Consider \( X = Y = \{a, b, c, d\} \) with topologies \( \tau = \{\emptyset, \{d\}, \{b, c\}, \{b, c, d\}, X\} \) and \( \sigma = \{\emptyset, \{b\}, \{b, c\}, \{b, c, d\}, Y\} \). Let \( f: X \to Y \) be the identity map. Then \( f \) is \( g^* \)-homeomorphism. But it is not regular homeomorphism. Since the inverse image of the closed set \( \{a\} \) in \( X \) is \( \{a\} \) is not closed in \( Y \).

**Theorem 5.7** Every \( g^* \)-homeomorphism is \( gpr \)-homeomorphism, but not conversely.

**Proof:** Let \( f: (X, \tau) \to (Y, \sigma) \) be a \( g^* \)-homeomorphism. Then \( f \) and \( f^{-1} \) are \( g^* \)-continuous and \( f \) is bijection. Since every \( g^* \)-continuous function is \( gpr \)-continuous, \( f \) and \( f^{-1} \) are \( gpr \)-continuous. Hence \( f \) is \( gpr \)-homeomorphism.

**Example 5.8** Consider \( X = Y = \{a, b, c, d\} \) with topology \( \tau = \{\emptyset, \{b\}, \{a, c\}, \{a, b, c\}, X\} \) and \( \sigma = \{\emptyset, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, Y\} \). Let \( f: X \to Y \) be a map defined by \( f(a) = b, f(b) = c, f(c) = d \) and \( f(d) = a \). Then \( f \) is \( gpr \)-homeomorphism. But it is not \( g^* \)-homeomorphism. Since the inverse image of the closed set \( \{d\} \) in \( X \) is \( \{c\} \) is not \( g^* \)-closed in \( Y \).

**Theorem 5.9** Every \( g^* \)-homeomorphism is \( rg \)-homeomorphism but not conversely.

**Proof:** The proof follows from the definition and fact that every \( g^* \)-closed set is \( rg \)-closed.

**Example 5.10** Consider \( X = Y = \{a, b, c, d\} \) with topology \( \tau = \{\emptyset, \{a, c\}, \{b, d\}, X\} \) and \( \sigma = \{\emptyset, \{d\}, \{a, b, c\}, Y\} \). Let \( f: X \to Y \) be the identity map. Then \( f \) is \( rg \)-homeomorphism. But it is not \( g^* \)-homeomorphism. Since the inverse image of the closed set \( \{a, c\} \) in \( X \) is \( \{a, c\} \) is not \( g^* \)-closed in \( Y \).

**Theorem 5.11** Every \( g^* \)-homeomorphism is \( g \)-homeomorphism but not conversely.

**Proof:** The proof follows from the definition and fact that every \( g^* \)-closed set is \( g \)-closed.

**Example 5.12** Consider \( X = Y = \{a, b, c, d\} \) with topology \( \tau = \{\emptyset, \{a, c\}, \{b, d\}, X\} \) and \( \sigma = \{\emptyset, \{d\}, \{a, b, c\}, Y\} \). Let \( f: X \to Y \) be the identity map. Then \( f \) is \( g \)-homeomorphism. But it is not \( g^* \)-homeomorphism. Since the inverse image of the closed set \( \{b, d\} \) in \( x \) is \( \{b, d\} \) is not \( g^* \)-closed in \( Y \).

**Remark 5.13** The composition of two \( g^* \)-homeomorphism need not be a \( g^* \)-homeomorphism in general as seen from the following example.

**Example 5.14** Let \( X = Y = Z = \{a, b, c\} \) with topologies \( \tau = \{\emptyset, \{b\}, X\}, \sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}, Y\} \) and \( \eta = \{\emptyset, \{a, b\}, Z\} \). Let \( g: (X, \tau) \to (Y, \sigma) \) be a map defined by \( g(a) = a, g(b) = c \) and \( g(c) = b \). \( f: (Z, \eta) \to (X, \tau) \) be the identity map. Both \( f \) and \( g \) are
gr*-homeomorphism. Define $g \circ f: (Z, \eta) \to (Y, \sigma)$. Hence $\{a\}$ is closed set of $(Y, \sigma)$. Therefore $(g \circ f)^{-1}\{a\} = \{a\}$ is not gr*-closed set of $(Z, \eta)$. Therefore $g \circ f$ is not gr*-homeomorphism.

**Remarks 5.15** For the functions defined above, we have the following implications.

### Theorem 5.16
Let $f: (X, \tau) \to (Y, \sigma)$ be a bijective gr*-continuous map. Then the following are equivalent.

(a) $f$ is an gr*-open map.
(b) $f$ is an gr*-homeomorphism.
(c) $f$ is an gr*-closed map.

**Proof:** Let $(X, \tau) \to (Y, \sigma)$ be a bijective gr*-continuous map.
(a)$\to$(b). Let $F$ be a closed set in $(X, \tau)$. Then $X/F$ is open in $(X, \tau)$. Since $f$ is gr*-open, then $f(X/F)$ is gr*-open in $(Y, \sigma)$. That is, $f(F)$ is gr*-closed in $(Y, \sigma)$. Thus $f$ is gr*-continuous. Further $(f^{-1})^{-1}(F) = f(F)$ is gr*-closed in $(Y, \sigma)$. Thus $f^{-1}$ is gr*-continuous.
(b)$\to$(c): Suppose $f$ is an gr*-homeomorphism. Then $f$ is bijective, $f$ and $f^{-1}$ are gr*-continuous. Let $S$ be a gr*-closed set in $(X, \tau)$. Since $f^{-1}$ is gr*-continuous. Then $(f^{-1})^{-1}(S) = f(S)$ is gr*-closed in $(Y, \sigma)$. Thus $f$ is gr*-closed.
(c)$\to$(a): Let $F$ be an gr*-closed map. Let $V$ be gr*-open in $X$. Then $X/V$ is gr*-closed in $(Y, \sigma)$. Since $f$ is gr*-closed, $f(X/V)$ is gr*-closed in $(Y, \sigma)$. This implies $Y/f(V)$ is gr*-closed in $(Y, \sigma)$. Therefore $f(V)$ is gr*-open in $(Y, \sigma)$.

### Definition 5.17
A bijection $f: (X, \tau) \to (Y, \sigma)$ is said to be gr**-homeomorphism if both $f$ and $f^{-1}$ are gr* irresolute. We say that spaces $(X, \tau)$ and $(Y, \sigma)$ are gr**-homeomorphism if there exists an gr**-homeomorphism from $(X, \tau)$ onto $(Y, \sigma)$. We denote the family of all gr**-homeomorphism of a topological space $(X, \tau)$ onto itself by gr**-h$(X, \tau)$.

### Theorem 5.18
Every gr**-homeomorphism is gr*-homeomorphism but not conversely.

**Proof:** Follows from the definition

### Example 5.19
Let $X = Y = \{a, b, c\}$, $\tau = \{\Phi, \{b\}, \{c\}, \{b, c\}, X\}$ and $\sigma = \{\Phi, \{b\} Y\}$. Define an identity map $f: (X, \tau) \to (Y, \sigma)$. Then $f$ is gr*-homeomorphism but not gr**-homeomorphism.

### Theorem 5.20
Let $f: (X, \tau) \to (Y, \sigma)$, and $g: (Y, \sigma) \to (Z, \eta)$ be gr**-homeomorphisms. Then their composition $g \circ f: (X, \tau) \to (Y, \sigma)$ is also gr**-homeomorphism.
Proof: Suppose \( f \) and \( g \) are \( g^{**}\)-homeomorphisms. Then \( f \) and \( g \) are \( g^*\)-irresolute. Let \( U \) be \( g^*\)-closed set in \((Z, \eta)\). Since \( g \) is \( g^*\)-irresolute, \( g^{-1}(U) \) is \( g^*\)-closed in \((Y, \sigma)\). This implies that \( f^{-1}(g^{-1}(U)) = (g\circ f)^{-1}(U) \) is \( g^*\)-closed in \((X, \tau)\). Since \( f \) is \( g^*\)-irresolute. Hence \((g\circ f)\) is \( g^*\)-irresolute. Also for an \( g^*\)-closed set \( V \) in \((X, \tau)\). We have \( g(f(V)) = g(f(V)) \). By hypothesis, \( f(V) \) is \( g^*\)-closed set in \((Z, \eta)\), this implies that \( g(f(V)) \) is \( g^*\)-closed set in \((Z, \eta)\). \( g\circ f \) is a bijection. This proves \( g\circ f \) is \( g^{**}\)-homeomorphism.

**Theorem 5.21** The set \( g^{**}(X, \tau) \) from \((X, \tau)\) onto itself is a group under composition of functions.

Proof: Let \( f \) and \( g \in g^{**}(X, \tau) \). Then by the theorem 5.20, \( g\circ f \in g^{**}(X, \tau) \). We know that the composition of functions is associative and the identity element. \( I: (X, \tau)\rightarrow (X, \tau) \) belonging to \( g^{**}(X, \tau) \) serves as the identity element. If \( f \in g^*(X, \tau) \), then \( f^{-1} \in g^*(X, \tau) \). This proves \( g^{**}(X, \tau) \) is a group under the operation of composition of functions.

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