On $W_2$-curvature tensor of a generalized complex space forms

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Abstract: In the present paper we study certain curvature conditions on $W_2$-curvature tensor. We study $W_2$-semisymmetric, $W_2$-flat generalized complex space forms. Also $W_2 \cdot S = 0$ and $W_2 \cdot R = 0$ on generalized complex space forms are studied.

Key words:- Generalized complex space forms, $W_2$-semisymmetric, $W_2$-flat.

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1. Introduction

In 1989 the author Olszak.Z.[7] has worked on existence of generalized complex space form. The authors U.C. De and A. Sarkar studied nature of a generalized Sasakian space form under some conditions regarding projective curvature tensor[8]. They also studied generalized Sasakian space forms with vanishing quasi-conformal curvature tensor and investigated quasi-conformal flat generalized Sasakian space form. The authors Venkatesha and B. Sumangala[10], Mehmet Atceken[6] Studied generalized space form satisfying certain conditions on M-projective curvature tensor and concircular curvature tensor. Motivated by these ideas, in this paper, we made an attempt to study $W_2$-curvature tensor in generalized complex space form.
2. Preliminaries

A Kaehler manifold is an even-dimensional manifold $M^{n_1}$, where $n_1 = 2k$ with a complex structure $J$ and a positive-definite metric $g$ which satisfies the following conditions [9].

\[ J^2 = -I, \]
\[ g(JX, JY) = g(X, Y) \text{ and} \]
\[ \nabla J = 0, \]

where $\nabla$ means covariant derivation according to the Levi-civita connection.

Let $(M, J, g)$ be a Kaehler manifold with constant holomorphic sectional curvature $c$. It is said to be a complex space form if the curvature tensor is of the form

\[ R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(Z, JY)JX - g(Z, JX)JY + g(X, JY)JZ]. \]  

(2.1)

An almost Hermitian manifold $M$ is called a generalized complex space form $M(f_1, f_2)$ if its Riemannian curvature tensor $R$ satisfies [4]

\[ R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ\}, \]

(2.2)

for all $X, Y, Z \in TM$ where $f_1$ and $f_2$ are smooth functions on $M$.

For generalized complex space form $M(f_1, f_2)$ we have

\[ S(X, Y) = \{(n_1 - 1)f_1 + 3f_2\}g(X, Y). \]  

(2.3)

\[ QX = [(n_1 - 1)f_1 + 3f_2]X. \]

(2.4)

\[ r = n_1[(n_1 - 1)f_1 + 3f_2], \text{ where } n_1 = 2k. \]

(2.5)

$S$ is the Ricci tensor, $Q$ is the Ricci operator and $r$ is scalar curvature of the space form $M(f_1, f_2)$. 

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Given an $n_1$ dimensional (where $n_1 = 2k$) a Kaehler manifold $M$. 

$W_2$ curvature tensor is given by

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n_1-1}[g(X,Z)QY - g(Y,Z)QX].$$

(2.6)

A $n_1$– dimensional generalized complex space form is said to be $W_2^-$ semisymmetric if it satisfies $R \cdot W_2 = 0$ where $R$ is the Riemannian curvature tensor of the space form.

**Theorem 2.1.** If $n_1$– dimensional generalized complex space form $M(f_1, f_2)$ satisfies $R \cdot W_2 = 0$ then it is either Ricci flat or $f_1(8n_1^3 - 12n_1^2 + 6n_1 - 1) - f_26n_1(n_1 - 1) + n_1 = 0$.

**proof:** Consider $R \cdot W_2 = 0$

$$(R(X,Y) \cdot W_2)(U,V,W) = 0.$$


Taking inner product with $Z$ we have

$$g(R(X,Y)W_2(U,V)W, Z) - g(W_2(R(X,Y)U,V)W, Z) - g(W_2(U,R(X,Y)V)W, Z) - g(W_2(U,V)R(X,Y)W, Z) = 0.$$

(2.7)

$$-g(W_2(U,V)R(X,Y)W, Z) = 0.$$

Using equations (2.2), (2.3), (2.4) and (2.6) in (2.7) and putting $X = V = Y = Z = e_i$ where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i(1 \leq i \leq n_1)$ we get after simplification that

$$\frac{f_1(8n_1^3 - 12n_1^2 + 6n_1 - 1) - f_26n_1(n_1 - 1) + n_1}{n_1(n_1 - 1)}S(U,W) = 0,$$

either $S(U,W) = 0$ or $f_1(8n_1^3 - 12n_1^2 + 6n_1 - 1) - f_26n_1(n_1 - 1) + n_1 = 0$.

This implies $M(f_1, f_2)$ is either Ricci flat or $f_1(8n_1^3 - 12n_1^2 + 6n_1 - 1) - f_26n_1(n_1 - 1) + n_1 = 0$.

**Theorem 2.2.** A $n_1$– dimensional generalized complex space form satisfying $W_2 = 0$ is an Einstein manifold.

**proof:** Suppose $W_2 = 0$ on generalized complex space form then from equation (2.6) we have
\[ R(X,Y)Z = -\frac{1}{n_1-1} [g(X,Z)QY - g(Y,Z)QX]. \]

*Using equation (2.4)* we have

\[ R(X,Y)Z = -\frac{1}{n_1-1} \{(n_1 - 1)f_1 + 3f_2\}[g(X,Z)Y - g(Y,Z)X]. \]

*Taking inner product with W* we have

\[ R(X,Y,Z,W) = -\frac{1}{n_1-1} \{(n_1 - 1)f_1 + 3f_2\}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]. \]

Putting \( Y = Z = e_i \) where \( \{e_i\} \) is an orthonormal basis of the tangent space at each point of the manifold and taking summation over \( i(1 \leq i \leq n_1) \) we get after simplification that

\[ S(X,W) = \frac{r}{n_1} g(X,W). \]

*Implies* \( M(f_1, f_2) \) *is an Einstein manifold.*

**Theorem 2.3.** A \( n_1 \)-dimensional \( (n_1 > 2) \) generalized complex space form \( M(f_1, f_2) \) satisfying \( W_2 \cdot S = 0 \) is either Einstein or \( (n_1 - 1)f_1 + 3f_2 = 0 \)

**proof:-** Consider \( W_2 \cdot S = 0.\)

\[ (2.8) \quad (W_2(X,Y)U,V) + S(U,W_2(X,Y)V) = 0. \]

*Using equation (2.3), (2.6) in equation (2.8) and putting \( Y = U = e_i \) where \( \{e_i\} \) is an orthonormal basis of the tangent space at each point of the manifold and taking summation over \( i(1 \leq i \leq n_1) \) we get*

\[ \frac{1}{n_1 - 1} \{(n_1 - 1)f_1 + 3f_2\}[(2-n_1)S(X,V) + rg(X,V)] = 0 \text{ where, } n_1 > 2. \]

*Implies either \( \{(n_1 - 1)f_1 + 3f_2\} = 0 \) or \( S(X,V) = \frac{1}{2-n_1} rg(X,V). \)

*Implies* \( M(f_1, f_2) \) *is an Einstein.*
Theorem 2.4. A $n_1$–dimensional generalized complex space form $M(f_1, f_2)$ satisfying $W_2 \cdot R = 0$ is an Einstein manifold.

**proof:-** Consider $W_2 \cdot R = 0$.

\[
\]

\[
g(W_2(X,Y)R(U,V)W,Z) - g(R(W_2(X,Y)U,V)W,Z) - g(R(U,W_2(X,Y)V)W,Z) - g(R(U,V)W_2(X,Y)W,Z) = 0.
\]  
(2.9)

Using equations (2.2), (2.6) in equation (2.9) and putting $X = Y = Z = e_i$ where \{\{e_i\}\} is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i(1 \leq i \leq n_1)$ we get after simplification that

\[
S(U,W) = \frac{(4n_1 - 1)f_1}{(-4n_1^2 + 6n_1 - 1)f_1} \cdot g(U,W).
\]

Implies $M(f_1, f_2)$ is an Einstein manifold.

**References**


