NEW MEIR-KEELER TYPE QUADRUPLE FIXED POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES

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Abstract: In this article, we prove a number of quadruple fixed point theorems by using a generalization of meir-keeler type contraction. We introduce an example to illustrate the effectiveness of our results. Also an application, some results of integral type are given.

Keywords: Quadruple fixed point; meir-keeler type contractions; partially ordered complete metric space.

1. INTRODUCTION


Following this trend, Berinde and Borcut [6] introduced the concept of triple fixed point and established some triple fixed point theorems in partially ordered metric spaces. The notion of fixed point of order $N \geq 3$ was first introduced by Samet and Vetro [7]. Hassen Aydi et al [15] defined generalized meir-keeler type functions and established some tripled fixed point theorems under a generalized meir-keeler contractive condition.

Very, recently; Karapinar [9] used the notion of quadruple fixed point and obtained some quadruple fixed point theorems in partially ordered metric spaces. Later, various results on quadruple fixed point have been obtained; see, for example, [9-14].

The purpose of this paper is three fold which can be described as follows.

1. We introduce the concept of mixed strict monotone property and generalized meir-keeler type functions and established a quadruple fixed point theorem for continuous mapping $F : X^4 \to X$ under a generalized Meir-keeler contractive condition in the setting of partially ordered metric spaces. We introduce an example to illustrate the effectiveness of our results.

2. We establish that quadruple fixed point theorem, is still valid for $F$, not necessarily continuous, assuming property $(X_\alpha)$ in $X$.

3. Also an application, some results of integral type are given.

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Here we recall some basic definitions and results.

The triple \((X, d, \leq)\) is called a partially ordered metric space if \((X, \leq)\) is a partially ordered set and \((X, d)\) is a metric space. Further if \((X, d)\) is complete metric space, then the triple \((X, d, \leq)\) is called a partially ordered complete metric space. Throughout the manuscript, we assume that \(X \neq \emptyset\) and

\[
X^k = X \times X \times \ldots \times X
\]

The mapping \(\rho_k: X^k \times X^k \to [0, \infty)\) such that

\[
\rho_k(x, y) := \max\{d(x_1, y_1), d(x_2, y_2), \ldots, d(x_k, y_k)\}
\]

forms a metric on \(X^k\), where \(x = (x_1, x_2, \ldots, x_k)\) and \(y = (y_1, y_2, \ldots, y_k) \in X^k\).

Definition 1.1(see [6]) Let \(X\) be a non empty set and let \(F: X^3 \to X\) be given mapping. An element \((x, y, z) \in X^3\) is called a tripled fixed point of \(F\) if

\[
F(x, y, z) = x, \quad F(y, z, y) = y, \quad F(z, x, y) = z.
\]

Definition 1.2 (see [6]) Let \((X, \leq)\) be a partially ordered set and \(F: X^3 \to X\) be a mapping. We say that \(F\) has the mixed monotone properly if \(F(x, y, z)\) is monotone nondecreasing in \(x\) and \(z\) and it is monotone non-increasing in \(y\), that is, for any \(x, y, z \in X\),

\[
x_1, x_2 \in X, \quad x_1 \leq x_2 \text{ implies } F(x_1, y, z) \leq F(x_2, y, z),
\]

\[
y_1, y_2 \in X, \quad y_1 \leq y_2 \text{ implies } F(x, y_1, z) \leq F(x, y_2, z),
\]

\[
z_1, z_2 \in X, \quad z_1 \leq z_2 \text{ implies } F(x, y, z_1) \leq F(x, y, z_2).
\]

Definition 1.3 (see [15]) Let \((X, \leq)\) be a partially ordered set and \(F: X^3 \to X\) be a mapping. We say that \(F\) has the mixed strict monotone property if \(x, y, z \in X\),

\[
x_1, x_2 \in X, \quad x_1 < x_2 \text{ implies } F(x_1, y, z) < F(x_2, y, z),
\]

\[
y_1, y_2 \in X, \quad y_1 < y_2 \text{ implies } F(x, y_1, z) < F(x, y_2, z),
\]

\[
z_1, z_2 \in X, \quad z_1 < z_2 \text{ implies } F(x, y, z_1) < F(x, y, z_2).
\]

Definition 1.4 (see [15]) Let \((X, d, \leq)\) be a partially ordered metric space. A mapping \(F: X^3 \to X\) is said to be generalized meir-keeler type contraction if for any \(\varepsilon > 0\), there exist a \(\delta(\varepsilon) > 0\) such that for all \(x, y, z, u, v, r \in X\) with \(x \leq u, y \geq v\) and \(z \leq r\),

\[
\varepsilon \leq \max\{d(x, u), d(y, v), d(z, r)\} < \varepsilon + \delta(\varepsilon)
\]

\[
\Rightarrow d(F(x, y, z), F(u, v, r)) < \varepsilon.
\]

Remark 1.1 (see [15]) It is immediate to show that if \(F: X^3 \to X\) is a generalized meir-keeler type contraction, then it is immediate to show that for all \(x, y, z, u, v, r \in X\) with \(x \leq u, y \geq v, z \leq r\) or \(x < u, y \geq v, z \leq r\),

\[
d(F(x, y, z), F(u, v, r)) < \max\{d(x, u), d(y, v), d(z, r)\}
\]
We consider the following partial order on the product space $X^4$: for all $(x, y, z, w), (u, v, h, l) \in X^4$

\[(1.8) \quad (x, y, z, w) \leq (u, v, h, l) \text{ if and only if } x \leq u, y \geq v, z \leq h, w \geq l.\]

We say that

\[(1.9) \quad (x, y, z, w) \text{ is equal to } (u, v, h, l) \text{ if and only if } x = u, y = v, z = h, w = l.\]

Also we say that $(x, y, z, w)$ and $(u, v, h, l)$ are comparable if

\[(1.10) \quad (x, y, z, w) \leq (u, v, h, l) \text{ or } (u, v, h, l) \leq (x, y, z, w).\]

**Definition 1.5** (see [9]) Let $X$ be a non empty set and let $F: X^4 \rightarrow X$ be given mapping. An element $(x, y, z, w) \in X^4$ is called a quadruple fixed point of $F$ if

\[(1.11) \quad F(x, y, z, w) = x, \quad F(y, z, w, x) = y, \quad F(z, w, x, y) = z, \quad F(w, x, y, z) = w.\]

Let $(X, d)$ be a metric space. The mapping $\bar{d}: X^4 \rightarrow X$ given by

\[(1.12) \quad \bar{d}(x, y, z, w)(u, v, h, l) = d(x, u) + d(y, v) + d(z, h) + d(w, l)\]

defines a metric on $X^4$, which will be denoted for convenience by $d$.

**Remark 1.2** In [9, 10, and 13] the notion of quadruple fixed point is called quartet fixed point.

**Definition 1.6** (see [9]) Let $(X, \preceq)$ be a partially ordered set and $F: X^4 \rightarrow X$ be a mapping. We say that $F$ has the mixed monotone property if $F(x, y, z, w)$ is monotone nondecreasing in $x$ and $z$ and it is monotone non-increasing in $y$ and $w$, that is, for any $x, y, z, w \in X$,

\[(1.13) \quad x_1, x_2 \in X, \quad x_1 \preceq x_2 \implies F(x_1, y, z, w) \leq F(x_2, y, z, w),\]

\[y_1, y_2 \in X, \quad y_1 \preceq y_2 \implies F(x, y_2, z, w) \leq F(x, y_1, z, w),\]

\[z_1, z_2 \in X, \quad z_1 \preceq z_2 \implies F(x, y, z_1, w) \leq F(x, y, z_2, w),\]

\[w_1, w_2 \in X, w_1 \preceq w_2 \implies F(x, y, z, w_2) \leq F(x, y, z, w_1).\]

### 2. MAIN RESULT

We introduce the following concepts.

**Definition 2.1** Let $(X, \preceq)$ be a partially ordered set and $F: X^4 \rightarrow X$ be a mapping. We say that $F$ has the mixed strict monotone property if $x, y, z, w \in X$.

\[(2.1) \quad x_1, x_2 \in X, x_1 < x_2 \implies F(x_1, y, z, w) < F(x_2, y, z, w),\]

\[y_1, y_2 \in X, y_1 < y_2 \implies F(x, y_2, z, w) < F(x, y_1, z, w),\]

\[z_1, z_2 \in X, z_1 < z_2 \implies F(x, y, z_1, w) < F(x, y, z_2, w),\]

\[w_1, w_2 \in X, w_1 < w_2 \implies F(x, y, z, w_2) < F(x, y, z, w_1).\]
Definition 2.2 Let $(X, d, \leq)$ be a partially ordered metric space. A mapping $F : X^4 \to X$ is said to be generalized meir-keeler type contraction if for any $\epsilon > 0$, there exist a $\delta(\epsilon) > 0$ such that for all $x, y, z, w, u, v, h, l \in X$ with $x \leq u, y \geq v, z \leq h$ and $w \geq l$,

\begin{equation}
\epsilon \leq \max\{d(x, u), d(y, v), d(z, h), d(w, l)\} < \epsilon + \delta
\end{equation}

\[ \Rightarrow d(F(x, y, z, w), F(u, v, h, l)) < \epsilon. \]

Remark 2.1 It is immediate to show that if $F : X^4 \to X$ is a generalized meir-keeler type contraction, then it is immediate to show that for all $x, y, z, w, u, v, h, l \in X$ with $x \leq u, y \geq v, z \leq h$ and $w \geq l$.

\begin{equation}
d(F(x, y, z, w), F(u, v, h, l)) < \max\{d(x, u), d(y, v), d(z, h), d(w, l)\}
\end{equation}

We introduce some notions used in the sequel. Let $\bar{F} : X^4 \to X^4$ such that for $a_1, a_2, a_3, a_4 \in X$.

\begin{equation}
\bar{F}(a_1, a_2, a_3, a_4) = \\
\{F(a_1, a_2, a_3, a_4), F(a_2, a_3, a_4, a_1), F(a_3, a_4, a_1, a_2), F(a_4, a_1, a_2, a_3)\}
\end{equation}

Let $x_0, y_0, z_0, w_0 \in X$ be such that

\begin{equation}
x_0 < F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0).
\end{equation}

\begin{equation}
z_0 < F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0).
\end{equation}

We’ll consider the four sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ such that

\begin{equation}
\begin{bmatrix}
x_n \\
y_n \\
z_n \\
w_n
\end{bmatrix} = \\
\begin{bmatrix}
F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) \\
F(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}) \\
F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}) \\
F(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1})
\end{bmatrix} + \\
\begin{bmatrix}
F^n(x_0, y_0, z_0, w_0) \\
F^n(y_0, z_0, w_0, x_0) \\
F^n(z_0, w_0, x_0, y_0) \\
F^n(w_0, x_0, y_0, z_0)
\end{bmatrix}, \quad \forall \ n \in \mathbb{N}.
\end{equation}

Our first auxiliary result is as follows.

Proposition 2.1 Let $(X, d, \leq)$ be a partially ordered metric space and $F : X^4 \to X$ be a mapping such that the following hypotheses hold:

1) $F$ has the mixed strict monotone property.
2) $F$ is a generalized meir-keeler type contraction,\n3) $\exists (x, y, z, w, (u, v, h, l) \in X^4$ such that $x < u, y \geq v, z < h, w \geq l$.

Then $\rho_4 \left(\bar{F}(x, y, z, w), \bar{F}(u, v, h, l)\right) \to 0$ as $n \to +\infty$.

Proof: Set $(x, y, z, w) = (x_0, y_0, z_0, w_0)$ and $(u, v, h, l) = (u_0, v_0, h_0, l_0)$. We assert that $\forall \ n \in \mathbb{N},$

\begin{equation}
x_n = F^n(x_0, y_0, z_0, w_0) < F^n(u_0, v_0, h_0, l_0) = u_n,
\end{equation}

\begin{equation}
y_n = F^n(y_0, z_0, w_0, x_0) > F^n(v_0, h_0, l_0, u_0) = v_n,
\end{equation}

\begin{equation}
z_n = F^n(z_0, w_0, x_0, y_0) < F^n(h_0, l_0, u_0, v_0) = h_n.
\end{equation}
\[ w_n = F^n(w_0, x_0, y_0, z_0) > F^n(l_0, u_0, v_0, h_0) = l_n, \]

with \( F = F^1 \).

Because \( F \) has the mixed strict monotone property, together with the assumption that \( x < u, y \geq v, z < h \) and \( w \geq l \), we obtain:

\[
x_1 = F(x, y, z) = F^1(x_0, y_0, z_0, w_0) < F(u_0, y_0, z_0, w_0)
\]

\[\Rightarrow F(x_0, y_0, z_0, w_0) < F(u_0, v_0, z_0, w_0)\]

\[\Rightarrow F(x_0, y_0, z_0, w_0) < F(u_0, v_0, h_0, w_0)\]

\[\Rightarrow F(x_0, y_0, z_0, w_0) < F(u_0, v_0, h_0, l_0) = u_1.\]

Analogously, we get:

\[
y_1 = F(y_0, z_0, w_0, x_0) > F(v_0, h_0, l_0, u_0) = v_1,
\]

\[
z_1 = F(z_0, w_0, x_0, y_0) < F(h_0, l_0, u_0, v_0) = h_1,
\]

\[
w_1 = F(w_0, x_0, y_0, z_0) > F(l_0, u_0, v_0, h_0) = l_1.
\]

Thus (2.7) holds for \( n = 1 \). By using the same arguments, we show that (2.7) holds also for \( n = 2 \). In fact,

\[
x_2 = F^2(x_0, y_0, z_0, w_0) = F(x_1, y_1, z_1, w_1)
\]

\[= F(F(x_0, y_0, z_0, w_0), F(y_0, z_0, w_0, x_0), F(z_0, w_0, x_0, y_0), F(w_0, x_0, y_0, z_0))\]

\[< F(F(u_0, v_0, h_0, l_0), F(v_0, h_0, l_0, u_0), F(z_0, w_0, x_0, y_0), F(w_0, x_0, y_0, z_0))\]

\[< F(F(u_0, v_0, h_0, l_0), F(v_0, h_0, l_0, u_0), F(z_0, w_0, x_0, y_0), F(w_0, x_0, y_0, z_0))\]

\[< F(F(u_0, v_0, h_0, l_0), F(v_0, h_0, l_0, u_0), F(h_0, l_0, u_0, v_0), F(w_0, x_0, y_0, z_0))\]

\[< F(F(u_0, v_0, h_0, l_0), F(v_0, h_0, l_0, u_0), F(h_0, l_0, u_0, v_0), F(l_0, u_0, v_0, h_0))\]

\[< F(F(u_0, v_0, h_0, l_0), F(v_0, h_0, l_0, u_0), F(h_0, l_0, u_0, v_0), F(l_0, u_0, v_0, h_0)) = u_2.\]

Analogously, we get:

\[
y_2 = F^2(y_0, z_0, w_0, x_0) > F^2(v_0, h_0, l_0, u_0) = v_2,
\]

\[
z_2 = F^2(z_0, w_0, x_0, y_0) < F^2(h_0, l_0, u_0, v_0) = h_2,
\]

\[
w_2 = F^2(w_0, x_0, y_0, z_0) > F^2(l_0, u_0, v_0, h_0) = l_2.
\]

Inductively, we get that (2.7) holds. Now by using Remark 2.1 and (2.7), we have;

\[
(2.8) \quad d(x_{n+2}, u_{n+2}) = d(F^{n+2}(x_0, y_0, z_0, w_0), F^{n+2}(u_0, v_0, h_0, l_0))
\]
\begin{align*}
d(y_{n+2}, v_{n+2}) &= d(F^{n+2}(y_0, z_0, x_0, y_0), F^{n+2}(v_0, h_0, l_0, u_0)) \\
&= d(F(y_{n+1}, z_{n+1}, u_{n+1}), F(v_{n+1}, h_{n+1}, l_{n+1}, u_{n+1})) \\
&< \max\{d(y_{n+1}, v_{n+1}), d(z_{n+1}, h_{n+1}), d(w_{n+1}, l_{n+1})\} \\
(2.9) \quad \text{Define Combining (2.8) - (2.11), we get;}

\begin{align*}
d(z_{n+2}, h_{n+2}) &= d(F^{n+2}(z_0, w_0, x_0, y_0), F^{n+2}(h_0, l_0, u_0, v_0)) \\
&= d(F(z_{n+1}, w_{n+1}, x_{n+1}, y_{n+1}), F(h_{n+1}, l_{n+1}, u_{n+1}, v_{n+1})) \\
&< \max\{d(z_{n+1}, h_{n+1}), d(w_{n+1}, l_{n+1}), d(x_{n+1}, u_{n+1})\} \\
(2.10) \quad \text{Consequently, the sequence is decreasing. Hence converges to, say } \eta. \\

\begin{align*}
d(w_{n+2}, l_{n+2}) &= d(F^{n+2}(w_0, x_0, y_0, z_0), F^{n+2}(l_0, u_0, v_0, h_0)) \\
&= d(F(w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}), F(l_{n+1}, u_{n+1}, v_{n+1}, h_{n+1})) \\
&< \max\{d(w_{n+1}, l_{n+1}), d(x_{n+1}, u_{n+1}), d(y_{n+1}, v_{n+1})\} \\
(2.11) \quad \text{Define } \Delta_{n+1} := \max\{d(x_{n+1}, u_{n+1}), d(y_{n+1}, v_{n+1}), d(z_{n+1}, h_{n+1}), d(w_{n+1}, l_{n+1})\} \\
\end{align*}

Combining (2.8) - (2.11), we get;}

\begin{align*}
\Delta_{n+2} < \Delta_{n+1}, \quad \forall \ n \in \mathbb{N} \\
\text{If we denote } B_n = (u_n, v_n, h_n, l_n) \text{, then by the definition of } \rho_4 \text{ and (2.13), we have}

\begin{align*}
\rho_4(A_{n+2}, B_{n+2}) < \rho_4(A_{n+1}, B_{n+1}), \quad \forall \ n \in \mathbb{N}. \\
(2.14) \quad \text{Consequently, the sequence } \{t_n\} = \{\rho_4(A_n, B_n)\} \text{ is decreasing. Hence } \{t_n\} \text{ converges to, say } \varepsilon \geq 0. \text{ Clearly, if } \varepsilon = 0, \text{ we have finished. Suppose, on the contrary, } \varepsilon > 0. \text{ Thus, there exists } k \in \mathbb{N} \text{ such that, for any } n \geq k.

\begin{align*}
\varepsilon \leq t_n = \rho_4(A_n, B_n) < \varepsilon + \delta(\varepsilon). \\
(2.15) \quad \text{In particular, for } n = k \text{ we have,}

\begin{align*}
\varepsilon \leq t_k = \rho_4(A_k, B_k) < \varepsilon + \delta(\varepsilon). \\
(2.16) \quad \text{That is,}

\begin{align*}
\varepsilon \leq \Delta_k < \varepsilon + \delta(\varepsilon). \\
(2.17) \quad \text{It follows from (2.7) and the hypothesis (ii) that}

\begin{align*}
d(F(x_k, y_k, z_k, w_k), F(u_k, v_k, h_k, l_k)) < \varepsilon. \\
(2.18)
\end{align*}
\end{align*}
\end{align*}
\end{align*}
This is equivalent to

\[(2.19) \quad d(x_{k+1}, u_{k+1}) < \varepsilon.\]

Analogously, we can get

\[(2.20) \quad d(y_{k+1}, v_{k+1}) < \varepsilon, \quad d(z_{k+1}, h_{k+1}) < \varepsilon, \quad d(w_{k+1}, l_{k+1}) < \varepsilon.\]

Combining (2.19) and (2.20) we have

\[(2.21) \quad \Delta_{k+1} < \varepsilon.\]

Thus, \(t_{k+1} = \rho_4(A_{k+1}, B_{k+1}) < \varepsilon.\) This is a contradiction and so \(\varepsilon = 0.\) We conclude that,

\[(2.22) \quad \rho_4(A_n, B_n) = \rho_4 \left( F^n(x, y, z, w), F^n(u, v, h, l) \right) \to 0 \text{ as } n \to +\infty.\]

Remark 2.2 The previous proposition remains true if in (iii), we change the assumption

\[\exists (x, y, z, w), (u, v, h, l) \in X^4 \text{ such that } x < u, y \geq v, z < h, w \geq l\]

with the following,

\[(2.23) \quad \exists (x, y, z, w), (u, v, h, l) \in X^4 \text{ such that } x \leq u, y > v, z \leq h, w > l.\]

3. EXISTENCE OF QUADRUPLE FIXED POINT

The following theorem is our main result.

Theorem 2.1 Let \((X, \leq, d)\) be a partially ordered metric space. \((X, d)\) be a complete metric space and \(F : X^4 \to X\) be a mapping such that the following hypotheses hold:

1) \(F\) is continuous,
2) \(F\) has the mixed strict monotone property,
3) \(F\) is a generalized meir-keeler type contraction,
4) There exist \(x_0, y_0, z_0, w_0 \in X\) such that

\[(2.24) \quad x_0 < F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0), \]

\[z_0 < F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0).\]

Then \(F\) has a quadruple fixed point that is, there exists \(x, y, z, w \in X\) such that

\[F(x, y, z, w) = x, \quad F(y, z, w, x) = y, \quad F(z, w, x, y) = z, \quad F(w, x, y, z) = w.\]

Proof: Let \(x_0, y_0, z_0, w_0 \in X\) be as in (2.24). We construct sequences \(\{x_n\}, \{y_n\}, \{z_n\}\) and \(\{w_n\}\) according to (2.7). We claim that for all \(n \geq 2,\)

\[(2.25) \quad x_n > x_{n-1}, \quad y_n < y_{n-1}, \quad z_n > z_{n-1}, \quad w_n < w_{n-1}.\]

Indeed, we will use a mathematical induction to prove (2.25). Clearly, we have
Suppose now that the inequalities in (2.25) hold for some \( n \geq 2 \). By the mixed strict monotone property of \( F \), together with (2.6), we have

\[
(2.27) \quad x_n = F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) < F(x_n, y_n, z_n, w_n) = x_{n+1},
\]

\[
y_n = F(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}) < F(y_n, z_n, w_n, x_n) = y_{n+1},
\]

\[
z_n = F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}) < F(z_n, w_n, x_n, y_n) = z_{n+1},
\]

\[
w_n = F(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}) > F(w_n, x_n, y_n, z_n) = w_{n+1}.
\]

Thus (2.25) holds for \( n \geq 2 \). Now putting \((x, y, z, w) = A_0, (u, v, h, l) = A_1 \) and Proposition 2.1, we get

\[
(2.28) \quad \rho_4 \left( \hat{F}(A_0), \tilde{F}(A_1) \right) \rightarrow 0 \text{ as } n \rightarrow +\infty.
\]

This is equivalent to

\[
(2.29) \quad \rho_4(A_n, A_{n+1}) \rightarrow 0 \text{ as } n \rightarrow +\infty.
\]

Take an arbitrary \( \varepsilon > 0 \), it follows from (2.29) that there exists \( k \in \mathbb{N} \) such that

\[
(2.30) \quad \rho_4(A_k, A_{k+1}) < \delta(\varepsilon).
\]

Without loss of generality, assume that \( \delta(\varepsilon) \leq \varepsilon \) and define the following set:

\[
(2.31) \quad \Pi := \{ A = (x, y, z, w) \in X^4 : \rho_4 \left( \hat{F}(A_k), \tilde{F}(A) \right) < \varepsilon + \delta(\varepsilon), x > x_k, y \leq y_k, z \geq z_k, w \leq w_k \}
\]

We claim:

\[
(2.32) \quad \hat{F}(A) \in \Pi, \forall A \in \Pi.
\]

Take \( A \in \Pi \), then by (2.30) and the triangle inequality, we have

\[
(2.33) \quad \rho_4 \left( (A_k), \tilde{F}(A) \right) = \max \{ d(x_k, F(x, y, z, w)), d(y_k, F(y, z, w, x)) \},
\]

\[
 \leq \max \{ d(x_k, x_{k+1}) + d(x_{k+1}, F(x, y, z, w)),
\]

\[
 d(y_k, y_{k+1}) + d(y_{k+1}, F(y, z, w, x)),
\]

\[
 d(z_k, z_{k+1}) + d(z_{k+1}, F(z, w, x, y)),
\]

\[
 d(w_k, w_{k+1}) + d(w_{k+1}, F(w, x, y, z)) \}
\]

\[
 \leq \max \{ d(x_k, x_{k+1}) + d(F(x_k, y_k, z_k, w_k), F(x, y, z, w)) \}.
\]
We consider the following two cases:

Case.1 \( \rho_4(A_k, A) \leq \varepsilon \). By Remark 2.1 and the definition of \( \prod \), the inequality (2.33) turns into

\[
\begin{align*}
\rho_4 \left( (A_k), \bar{F}(A) \right) &< \delta(\varepsilon) + \rho_4 \left( \bar{F}(A_k), \bar{F}(A) \right) \\
&< \delta(\varepsilon) + \rho_4(A_k, A) < \delta(\varepsilon) + \varepsilon
\end{align*}
\]

Case.2 If \( \varepsilon < \rho_4(A_k, A) < \delta(\varepsilon) + \varepsilon \), that is

\[
\varepsilon < \max\{d(x, x_k), d(y, y_k), d(z, z_k), d(w, w_k)\} < \delta(\varepsilon) + \varepsilon
\]

Since \( x > x_k, y \leq y_k, z > z_k, w \leq w_k \), then by hypothesis (3) we have

\[
\begin{align*}
d(F(x, y, z, w), F(x_k, y_k, z_k, w_k)) &< \varepsilon, \\
d(F(y, z, w, x), F(y_k, z_k, w_k, x_k)) &< \varepsilon, \\
d(F(z, w, x, y), F(z_k, w_k, x_k, y_k)) &< \varepsilon, \\
d(F(w, x, y, z), F(w_k, x_k, y_k, z_k)) &< \varepsilon,
\end{align*}
\]

Hence, combining (2.36)-(2.39) and (2.33) we get

\[
\rho_4 \left( (A_k), \bar{F}(A) \right) < \delta(\varepsilon) + \varepsilon
\]

On other hand using (2), one can easily check that

\[
F(x, y, z, w) > x_k, \quad F(y, z, w, x) \leq y_k, \quad F(z, w, x, y) > z_k, \quad F(w, x, y, z) \leq w_k.
\]

Hence, we conclude that (2.32) holds by (2.31), we have that \( A_{k+1} \in \prod \) and so by (2.32), we get

\[
\bar{F}(A_{k+1}) = A_{k+2} \in \prod
\Rightarrow \bar{F}(A_{k+2}) = A_{k+3} \in \prod
\vdots
\Rightarrow \bar{F}(A_n) \in \prod, \quad \forall \ n \geq k
\]
Then, for all $n, m > k$ we have,

\[(2.43) \quad \rho_4(A_n, A_m) \leq \rho_4(A_n, A_k) + \rho_4(A_k, A_m) < 2(\varepsilon + \delta(\varepsilon)) \leq 4\varepsilon\]

Hence $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ are Cauchy sequences in the metric space $(X^4, \rho_4)$. Since $(X, d)$ is a complete, then $(X^4, \rho_4)$ is also complete. Then, there exists a point $(x, y, z, w) \in X^4$.

\[(2.44) \quad d(x_n, x), d(y_n, y), d(z_n, z), d(w_n, w) \to 0 \text{ as } n \to +\infty.\]

Since $F$ is continuous and $(x_n, y_n, z_n, w_n) \to (x, y, z, w)$, we have $x_{n+1} = F(x_n, y_n, z_n, w_n) \to (x, y, z, w)$. By the uniqueness of limit, we get that $x = F(x, y, z, w)$. Similarly, we can show that

\[y = F(y, z, w, x), z = F(z, w, x, y), w = F(w, x, y, z).\]

So $(x, y, z, w)$ is a quadruple fixed point of $F$. This finishes the proof.

Remark 2.3: Theorems remains true if we replace (4) with one of the following statements. There exists $x_0, y_0, z_0, w_0 \in X$ such that,

1) $x_0 \leq F(x_0, y_0, z_0, w_0), \quad y_0 > F(y_0, z_0, w_0, x_0),
   z_0 \leq F(z_0, w_0, x_0, y_0), \quad w_0 > F(w_0, x_0, y_0, z_0).$

2) $x_0 \leq F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0),
   z_0 < F(z_0, w_0, x_0, y_0), \quad w_0 > F(w_0, x_0, y_0, z_0).$

3) $x_0 < F(x_0, y_0, z_0, w_0), \quad y_0 > F(y_0, z_0, w_0, x_0),
   z_0 \leq F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0).$

4) $x_0 < F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0),
   z_0 < F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0).$

5) $x_0 \leq F(x_0, y_0, z_0, w_0), \quad y_0 > F(y_0, z_0, w_0, x_0),
   z_0 < F(z_0, w_0, x_0, y_0), \quad w_0 > F(w_0, x_0, y_0, z_0).$

6) $x_0 < F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0),
   z_0 < F(z_0, w_0, x_0, y_0), \quad w_0 > F(w_0, x_0, y_0, z_0).$

7) $x_0 < F(x_0, y_0, z_0, w_0), \quad y_0 > F(y_0, z_0, w_0, x_0),
   z_0 < F(z_0, w_0, x_0, y_0), \quad w_0 > F(w_0, x_0, y_0, z_0).$

8) $x_0 \leq F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0),
   z_0 < F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0).$

9) $x_0 \leq F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0)
   z_0 \leq F(z_0, w_0, x_0, y_0), \quad w_0 > F(w_0, x_0, y_0, z_0).$

10) $x_0 < F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0),
    z_0 \leq F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0).$

11) $x_0 \leq F(x_0, y_0, z_0, w_0), \quad y_0 > F(y_0, z_0, w_0, x_0),
    z_0 \leq F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0).$

12) $x_0 \leq F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0),
    z_0 \leq F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0).$

13) $x_0 \leq F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0),
    z_0 \leq F(z_0, w_0, x_0, y_0), \quad w_0 > F(w_0, x_0, y_0, z_0).$
The following examples illustrates theorem 2.1

Example 2.1 Let $X = \mathbb{R}$ with the metric $d(x, y) = |x - y|$ and the usual ordering. Clearly $(X, d, \leq)$ is a partially ordered complete metric space. Let $F: X^4 \to X$ be defined by

$$F(x, y, z, w) = \frac{2x-y+2z-w+2}{10}$$

for all $x, y, z, w \in X$. Obviously, $F$ is continuous and has the mixed strict monotone property. Moreover taking $x_0 = 0, y = \frac{1}{2}, z = 0, w = \frac{1}{2}$, we have:

$$F(x_0, y_0, z_0, w_0) = \frac{1}{10}, \quad F(y_0, z_0, w_0, x_0) = \frac{2}{5}$$

and then, condition (4) of Theorem 2.1 holds for $(x_0, y_0, z_0, w_0) = \left(0, \frac{1}{2}, 0, \frac{1}{2}\right)$. On the other hand, for $x, y, z, w, u, v, h, l \in X$ with $x < u, y \geq v, z < h, w \geq l$ we have

$$\varepsilon \leq \max\{d(x, u), d(y, v), d(z, h), d(w, l)\}$$

$$= \max\{u - x, v - y, h - z, l - w\} < \varepsilon + \delta(\varepsilon)$$

$$d(F(x, y, z, w), (u, v, h, l)) = \left|\frac{2x-y+2z-w+2}{10} - \frac{2u-v+2h-l+2}{10}\right|$$

$$\leq \frac{u-x}{5} + \frac{v-y}{10} + \frac{h-z}{5} + \frac{l-w}{10}$$

Then condition (4) of theorem (2.1) hold for $\delta(\varepsilon) < \frac{2}{3} \varepsilon$. That is, $F$ is a generalized meir-keeler type contraction. It is clear that all the hypotheses of Theorem 2.1 are satisfies and in view of Theorem 2.1, $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is the desired quadruple fixed point of $F$.

In what follows, we prove that theorem 2.1, is still valid for $F$, not necessarily continuous, assuming property $(X_p)$ in $X$.

$(X_p)$: $X$ has the following properties:

a) If $\{x_n\}$ is a sequence in $X$ such that $x_{n+1} > x_n, \forall \ n \in \mathbb{N}$ and $x_n \to x$, then $x_n < x, \forall \ n \in \mathbb{N}$.

b) If $\{y_n\}$ is a sequence in $X$ such that $y_{n+1} < y_n, \forall \ n \in \mathbb{N}$ and $y_n \to y$, then $y_n > x, \forall \ n \in \mathbb{N}$.

Theorem: 2.2 Let $(X, d, \leq)$ be a partially ordered complete metric space. Suppose $X$ has the property $(X_p)$. Assume that $F: X^4 \to X$ satisfies the following hypothesis.

1. $F$ has the mixed strict monotone property,
2. $F$ is a generalized meir-keeler type contraction,
3. There exists $x_0, y_0, z_0, w_0 \in X$ such that

$$x_0 < F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0),$$

$$z_0 < F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0)$$
Then $F$ has a quadruple Fixed point, that is there exists $x, y, z, w \in X$ such that

$$ F(x, y, z, w) = x, F(y, z, w, x) = y, F(z, w, x, y) = z, F(w, x, y, z) = w. $$

Proof: Following the same lines of the proof of the Theorem 2.1, we remain to prove that

$$ F(x, y, z, w) = x, F(y, z, w, x) = y, F(z, w, x, y) = z, F(w, x, y, z) = w. $$

To this aim, take an arbitrary $\varepsilon > 0$. since

$$ x_n = F^n(x_0, y_0, z_0, w_0) \to x, \quad y_n = F^n(y_0, z_0, w_0, x_0) \to y, $$

$$ z_n = F^n(z_0, w_0, x_0, y_0) \to z, \quad w_n = F^n(w_0, x_0, y_0, z_0) \to w. $$

Then there exists $n_1, n_2, n_3, n_4 \in \mathbb{N}$ such that

$$ d(x_p, x) = d(F^n(x_0, y_0, z_0, w_0), x) < \varepsilon, $$

$$ d(y_q, y) = d(F^n(y_0, z_0, w_0, x_0), y) < \varepsilon, $$

$$ d(z_r, z) = d(F^n(z_0, w_0, x_0, y_0), z) < \varepsilon, $$

$$ d(w_s, w) = d(F^n(w_0, x_0, y_0, z_0), w) < \varepsilon. $$

for all $p \geq n_1, q \geq n_2, r \geq n_3, s \geq n_4$. Now taking $n = \max_{1 \leq i \leq 2}\{n_i\}$ and using Remark 2.1 with the assumption,

$$ x_n = F^n(x_0, y_0, z_0, w_0) < x, \quad y_n = F^n(y_0, z_0, w_0, x_0) > y, $$

$$ z_n = F^n(z_0, w_0, x_0, y_0) < z, \quad w_n = F^n(w_0, x_0, y_0, z_0) > w. $$

From (2.44) we get

$$ d(x, F(x, y, z, w)) \leq d(x, x_{n+1}) + d(x_{n+1}, F(x, y, z, w)) $$

$$ = d(x, x_{n+1}) + d(F(x_{n+1}, y_0, z_0, w_0), F(x, y, z, w)) $$

$$ = d(x, x_{n+1}) + d(F(x_n, y_n, z_n, w_n), F(x, y, z, w)) $$

$$ = d(x, x_{n+1}) + \max\{d(x_n, x), d(y_n, y), d(z_n, z), d(w_n, w)\} $$

$$ < 2\varepsilon. $$

Analogously, we get that

$$ d(y, F(y, z, w, x)) < 2\varepsilon, \quad d(z, F(z, w, x, y)) < 2\varepsilon, \quad d(w, F(w, x, y, z)) < 2\varepsilon. $$

This yield that

$$ F(x, y, z, w) = x, F(y, z, w, x) = y, F(z, w, x, y) = z, F(w, x, y, z) = w. $$

4. **UNIQUENESS OF QUADRUPLE FIXED POINT**
In this section, we will prove the uniqueness of the quadruple fixed point.

**Theorem 2.3** In addition to the hypotheses of the Theorem 2.1, assume that for all \((x, y, z, w), (u, v, h, l) \in X^4\), there exists \((a, b, c, d) \in X^4\), such that \((a, b, c, d)\) comparable to \(F(x, y, z, w)\) and \((u, v, h, l)\). Then \(F\) has a unique quadruple fixed point.

Proof: The set of quadruple fixed point of \(F\) is not empty due to Theorem 2.1. Assume now, \(A = (x, y, z, w)\) and \(A^* = (x^*, y^*, z^*, w^*) \in X^4\) are two quadruple fixed point of \(F\), that is

\[
F(x, y, z, w) = x, \quad F(y, z, w, x) = y,
\]

\[
F(z, w, x, y) = z, \quad F(w, x, y, z) = w,
\]

\[
F(x^*, y^*, z^*, w^*) = x^*, F(y^*, z^*, w^*, x^*) = y^*,
\]

\[
F(z^*, w^*, x^*, y^*) = z^*, F(w^*, x^*, y^*, z^*) = w^*.
\]

We shall show that \(A = A^*\). We distinguish the following two cases.

**Case 1** If \((x, y, z, w)\) is comparable to \((x^*, y^*, z^*, w^*)\) with respect to the ordering in \(X^4\), where

\[
\lim_{n \to +\infty} F^n(x_0, y_0, z_0, w_0) = x,
\]

\[
\lim_{n \to +\infty} F^n(y_0, z_0, w_0, x_0) = y,
\]

\[
\lim_{n \to +\infty} F^n(z_0, w_0, x_0, y_0) = z,
\]

\[
\lim_{n \to +\infty} F^n(w_0, x_0, y_0, z_0) = w.
\]

Without loss of generality, we may assume that,

\[
x = F(x, y, z, w) < F(x^*, y^*, z^*, w^*) = x^*,
\]

\[
y = F(y, z, w, x) \geq F(y^*, z^*, w^*, x^*) = y^*,
\]

\[
z = F(z, w, x, y) < F(z^*, w^*, x^*, y^*) = z^*,
\]

\[
w = F(w, x, y, z) \geq F(w^*, x^*, y^*, z^*) = w^*.
\]

By the definition of \(p_4\) and Remark 2.1, we have

\[
p_4(A, A^*) = p_4((x, y, z, w), (x^*, y^*, z^*, w^*))
\]

\[
= \max\{d(x, x^*), d(y, y^*), d(z, z^*), d(w, w^*)\}
\]

\[
= \max\{d(F(x, y, z, w), F(x^*, y^*, z^*, w^*)), d(F(y, z, w, x), F(y^*, z^*, w^*, x^*)),
\]

\[
d(F(z, w, x, y), F(z^*, w^*, x^*, y^*)), d(F(w, x, y, z), F(w^*, x^*, y^*, z^*))\}
\]

\[
< \max\{d(x, x^*), d(y, y^*), d(z, z^*), d(w, w^*)\}
\]

\[
= p_4((x, y, z, w), (x^*, y^*, z^*, w^*)) = p_4(A, A^*).
\]
This is a contradiction, therefore must be $A = A^*$. 

Case 2: If $(x, y, z, w)$ is not comparable to $(x^*, y^*, z^*, w^*)$. By assumption there exists $B = (a, b, c, d) \in X^4$ which is comparable to both $A$ and $A^*$. Without loss of generality, we may assume that

\begin{align*}
x = F(x, y, z, w) &< a, \quad y = F(y, z, w, x) \geq b, \\
z = F(z, w, x, y) &< c, \quad w = F(w, x, y, z) \geq d.
\end{align*}

and

\begin{align*}
x^* = F(x^*, y^*, z^*, w^*) &< a, \quad y^* = F(y^*, z^*, w^*, x^*) \geq b, \\
z^* = F(z^*, w^*, x^*, y^*) &< c, \quad w^* = F(w^*, x^*, y^*, z^*) \geq d.
\end{align*}

By Proposition 2.1, and using (2.47), (2.48) we have

\begin{equation}
\rho_4 \left( \overline{F^4}(A), \overline{F^4}(B) \right) \rightarrow 0 \text{ as } n \rightarrow +\infty.
\end{equation}

and

\begin{equation}
\rho_4 \left( \overline{F^4}(A^*), \overline{F^4}(B) \right) \rightarrow 0 \text{ as } n \rightarrow +\infty.
\end{equation}

By the triangle inequality, we get

\begin{equation}
\rho_4 (A, A^*) = \rho_4 \left( \overline{F^4}(A), \overline{F^4}(A^*) \right) \\
\leq \rho_4 \left( \overline{F^4}(A^*), \overline{F^4}(B) \right) + \rho_4 \left( \overline{F^4}(B), \overline{F^4}(A^*) \right) \rightarrow 0 \text{ as } n \rightarrow +\infty.
\end{equation}

This implies that $A = A^*$. 

Corollary 2.1 In addition to the hypotheses of Theorem 2.2 assume that for all $(x, y, z, w), (u, v, h, l) \in X^4$ there exists $(a, b, c, d) \in X^4$ that is comparable to $(x, y, z, w)$ and $(u, v, h, l)$. Then $F$ has a unique quadruple fixed point.

Remark: 2.4 In view of theorem 2.2, the mapping $F$ in example 2.1 has a unique quadruple fixed point.

5. RESULTS OF INTEGRAL TYPE

Motivated by Suzuki [16] and on the same lines of theorem 3.11 of [8], one can prove the following result.

Theorem: 2.4 Let $(X, d, \leq)$ be a partially ordered complete metric space and $F: X^4 \rightarrow X$ be a given mapping. Assume that there exists a function $\theta$ from $[0, +\infty)$ into itself satisfying the following:

1. $\theta(0) = 0$ and $\theta(t) > 0$ for every $t > 0$.
2. $\theta$ is a non decreasing and right continuous.
3. For every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

\begin{equation}
\varepsilon \leq \theta(\max\{d(x, u), d(y, v), d(z, h), d(w, l)\}) < \varepsilon + \delta(\varepsilon)
\end{equation}

\[ \Rightarrow \theta(d(F(x, y, z, w), F(u, v, h, l))) < \varepsilon. \]
for all \( x \geq u, y \leq v, z \geq h, w \leq l \). Then \( F \) is a generalized meir-keeler type contraction.

Corollary 2.2 Let \((X, d, \leq)\) be a partially ordered complete metric space and \( F: X^4 \rightarrow X \) be a given mapping satisfying the following hypotheses:

1. \( F \) has the mixed strict monotone property.
2. For every \( \varepsilon > 0 \), there exists \( \delta(\varepsilon) > 0 \) such that
   \[
   (2.58) \quad \varepsilon \leq \int_0^{\max\{d(x,u),d(y,v),d(z,h),d(w,l)\}} \varphi(t)dt < \varepsilon + \delta(\varepsilon)
   \]
   \[
   \Rightarrow \int_0^{d(F(x,y,z,w),F(u,v,h,l))} \varphi(t)dt < \varepsilon.
   \]
   for all \( x \geq u, y \leq v, z \geq h, w \leq l \), where \( \varphi: [0, +\infty) \rightarrow [0, +\infty) \) is a locally integrable function satisfying
   \[
   (2.59) \quad \int_0^s \varphi(t)dt > 0 \text{ for all } s > 0.
   \]
3. There exists \( x_0, y_0, z_0, w_0 \in X \) such that
   \[
   x_0 < F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0),
   \]
   \[
   z_0 < F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0).
   \]
   Assume, either \( F \) is continuous or property \((X_p)\) given in Theorem 2.2 hold. Then \( F \) has a quadruple fixed point.

To end this paper, we give the following corollary.

Corollary 2.3 Let \((X, d, \leq)\) be a partially ordered complete metric space and \( F: X^4 \rightarrow X \) be a given mapping satisfying the following hypotheses:

1. \( F \) has the mixed strict monotone property.
2. For all \( x \geq u, y \leq v, z \geq h, w \leq l \),
   \[
   (2.60) \quad \int_0^{d(F(x,y,z,w),F(u,v,h,l))} \varphi(t)dt \leq k \int_0^{\max\{d(x,u),d(y,v),d(z,h),d(w,l)\}} \varphi(t)dt
   \]
   where \( k \in (0,1) \) and \( \varphi: [0, +\infty) \rightarrow [0, +\infty) \) is a locally integrable function satisfying
   \[
   (2.61) \quad \int_0^s \varphi(t)dt > 0 \text{ for all } s > 0.
   \]
3. There exists \( x_0, y_0, z_0, w_0 \in X \) such that
   \[
   x_0 < F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0),
   \]
   \[
   z_0 < F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0).
   \]
   Assume, either \( F \) is continuous or property \((X_p)\) given in Theorem 2.2 hold. Then \( F \) has a quadruple fixed point.

6. COMPETING INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.
7. AUTHOR'S CONTRIBUTION

All authors contributed equally and significantly to writing this paper. All authors read and approved the final paper.

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