ON THE HYPER-WIENER INDEX OF THORNY-COMPLETE GRAPH

Shigehalli V. S. 1 and Shanmukh Kuchabal 2

1 Professor, Department of Mathematics, Rani Channamma University,
Vidya Sangama, Belagavi-591156, India.

2 Research Scholar, Department of Mathematics, Rani Channamma University,
Vidya Sangama, Belagavi-591156, India.

Email: shanmukhkuchabal@gmail.com, shigehallivs@yahoo.co.in

Abstract: Let G be the graph. The Wiener Index W(G) is the sum of all distances between vertices of G, where as the Hyper-Wiener index WW(G) is defined as $\text{WW}(G) = \text{W}(G) + \frac{1}{2} \sum_{(u,v) \in V(G)} d^2(u,v)$. In this paper we prove some general results on Hyper-Wiener Index of Thorny-Complete graphs.

Mathematics Subject Classification: 05C12.

Keywords: Thorny-complete graph, Wiener index and hyper-Wiener index.

1. INTRODUCTION:

In this paper we consider graphs means simple connected graphs, connected graphs without loops and multiple edges. In mathematical terms a graph is represented as $G = (V,E)$ where $V$ is the set of vertices and $E$ is the set of edges. The distance between the vertices $u$ and $v$ of $V(G)$ is denoted by $d(u,v)$ and it is defined as the number of edges in a minimal path connecting the vertices $u$ and $v$.

In chemical graph theory, the Wiener index (also called Wiener number) is a topological index of a molecule, defined as the sum of the lengths of the shortest paths between all pairs of vertices in the chemical graph represented the non-hydrogen atom in the molecule. The Wiener index is named after Harry Wiener, who introduced it in 1947; at the time, Wiener called it the “Path Number”. It is the oldest topological index related to molecular branching.

Wiener Index given by

$$ W(G) = \sum_{u \in V} d(u,v) $$

The Hyper-Wiener index “WW” is distance based graph invariants, used as a structure descriptor for predicting physico-chemical properties of organic compounds. The hyper-Wiener index of acyclic graphs was introduced by Milan Randic in 1993. Then Klein, generalized Randic’s definition for all connected graphs, as a generalization of the Wiener index. It is defined as

$$ \text{WW}(G) = \text{W}(G) + \frac{1}{2} \sum_{(u,v) \in V(G)} d^2(u,v) $$

The Hyper-Wiener index of Complete graph $K_n$, Path graph $P_n$, Star graph $K_{1,n-1}$ and Cycle graph $C_n$ is given by the expressions

$$ \text{WW}(K_n) = \frac{n(n-1)}{2}, \quad \text{WW}(P_n) = \frac{n^4+2n^3-n^2-2n}{24}, \quad \text{WW}(K_{1,n-1}) = \frac{1}{2} (n-1)(3n-4) $$

$$ \text{WW}(C_n) = \begin{cases} 
\frac{n^2(n+1)(n+2)}{48}, & \text{n is even} 
\end{cases} $$
We have three methods for calculation of the Hyper-Wiener Index of molecular graphs.

(i) **Distance Formula:**

\[ WW(G) = W(G) + \frac{1}{2} \sum_{(u,v) \in V(G)} d^2(u,v) \]

(ii) **Cut Method:**

(iii) **The Method of Hosaya Polynomials:**

Let \( G \) a connected \( n \)-vertex graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and \( p = (p_1, p_2, \ldots, p_n) \) be an \( n \)-tuple of non-negative integers. The **thorn graph** \( G_p \) is the graph obtained by attaching \( p_i \) pendent vertices to the vertex \( v_i \) of \( G \) for \( i = 1, 2, \ldots, n \). The \( p_i \) pendent vertices attached to the vertex \( v_i \) will be called the thorns of \( v_i \). The concept of thorny graphs was introduced by Ivan Gutman.

Following results are proved using distance formula

### 2. MAIN RESULTS:

**Theorem 2.1:** Let \( K_n \) be the compete graph on \( n \) vertices. The graph \( G \) obtained by attaching \( S \)-number of Pendent vertices to each vertex of \( K_n \) with common vertex then its Hyper-Wiener index given by

\[ WW(G) = \frac{1}{2} \left\{ n^2 - n - 25n + 6Pn - 5P - 3PS + 6P^2 \right\} \]

Where

- \( n \) -cardinality of complete graph
- \( S \) - Number of Pendent vertices attached to each vertex of \( n \).
- \( P \) – Total number of Pendent vertices present in \( G \).

**Proof:** To find Hyper-Wiener index of the graph we need to find following two parts,

To find \( W(G) \):

\[
W(G) = \frac{1}{2} \sum_{(u,v) \in V(G)} d(u,v) \\
= \frac{1}{2} [(1 + 1 + \cdots + 1 + 2 + 2 + \cdots + 2) + \cdots + (1 + 1 + \cdots + 1 + 2 + 2 + \cdots + 2)] \\
\quad \times \left( \begin{array}{c} \text{n + S - 1 times} \\ \text{P-S times} \\
\text{n + S - 1 times} \\
\text{P-S times} \\
\text{n + S - 2 times} \\
\text{P-S times} \\
\text{n + S - 1 times} \\
\text{P-S times} \\
\end{array} \right)
\]

\[
= \frac{1}{2} [(n + S - 1 + 2(P - S) + \cdots + n + S - 1 + 2(P - S)]
\]

\[
\times \left( \begin{array}{c} n \text{ times} \\
\text{P times} \\
\end{array} \right) \\
+ [1 + 2(n + S - 2) + 3(P - S) + \cdots + 1 + 2(n + S - 2) + 3(P - S)]
\]

\[
\times \left( \begin{array}{c} \text{n times} \\
\text{P times} \\
\end{array} \right)
\]
\[ W(G) = \frac{1}{2} [n^2 - 5n - n + 4Pn - 3P - PS + 3P^2] \] 
\[ \text{...........................................(a)} \]

To find \( W^*(G) \):
\[
W^*(G) = \frac{1}{2} \sum_{(u,v) \in V(G)} d^2(u,v)
\]
\[
W^*(G) = \frac{1}{2} \left[ ((1 + 1 + \cdots + 1) + (1 + 1 + \cdots + 1) + \cdots + (1 + 1 + \cdots + 1)) \right]
\]
\[
\text{P-S times P-S times P-S times}
\]
\[
+ (1 + 1 + \cdots + 1 + 3 + 3 + \cdots + 3) + \cdots + (1 + 1 + \cdots + 1 + 3 + 3 + \cdots + 3)
\]
\[
\text{n + S - 2 times P-S times n + S - 2 times P-S times}
\]
\[
= \frac{1}{2} \left[ ((P - S) + (P - S) + \cdots + (P - S)) \right]
\]
\[
\text{n times}
\]
\[
+ ((n + S - 2) + 3(P - S) + \cdots + (n + S - 2) + 3(P - S)) \right]
\]
\[
\text{P times}
\]
\[
W^*(G) = \frac{1}{2} \left[ n(P - S) + P((n + S - 2) + 3(P - S)) \right]
\]
\[
W^*(G) = \frac{1}{2} \left[ 2Pn - Sn - 2PS - 2P + 3P^2 \right] \] 
\[ \text{...........................................(b)} \]

Since \( W(G) = W(G) + W^*(G) \)

Therefore \( W(G) = \frac{1}{2} [n^2 - 5n - n + 4Pn - 3P - PS + 3P^2] \)
\[
+ \frac{1}{2} \left[ 2Pn - Sn - 2PS - 2P + 3P^2 \right]
\]

Combining (a) and (b) gives
\[
W(G) = \frac{1}{2} \left[ n^2 - n - 2Sn + 6Pn - 5P - 3PS + 6P^2 \right]
\]

Corollary 2.1.1: Let \( K_n \) be the compete graph on \( n \) vertices. The graph \( G \) obtained by attaching three Pendent vertices to each vertex of \( K_n \) with common vertex then its Hyper-Wiener index given by
\[
W(G) = \frac{1}{2} \left[ n^2 - 7n + 6Pn - 14P + 6P^2 \right]
\]

Proof: Substituting \( S=3 \) in above theorem, gives the result. 

Corollary 2.1.2: Let \( K_n \) be the compete graph on \( n \) vertices. The graph \( G \) obtained by attaching two Pendent vertices to each vertex of \( K_n \) with common vertex then its Hyper-Wiener index given by
WW(G) = \frac{1}{2}\{n^2 - 5n + 6Pn - 11P + 6P^2\}

**Proof:** Substituting S=2 in above theorem, gives the result.

**Corollary 2.1.3:** Let $K_n$ be the compete graph on n vertices. The graph G obtained by attaching one Pendent vertex to each vertex of $K_n$ with common vertex then its Hyper-Wiener index given by

$$WW(G) = \frac{1}{2}\{n^2 - 3n + 6Pn - 8P + 6P^2\}$$

**Proof:** Substituting S=1 in above theorem, gives the result.

**Corollary 2.1.4:** Hyper-Wiener index of complete graph given by

$$WW(G) = \frac{1}{2}\{n^2 - n\}$$

**Proof:** Substituting S=0 and P=0 in above theorem, gives the result.

**Illustrations:**

\[\text{Theorem 2.2:} \quad \text{Let } K_n \text{ be the compete graph on } n \text{ vertices (n is even). The graph G obtained by attaching S- number of Pendent vertices to alternative vertices of } K_n \text{ with common vertex then its Hyper-Wiener index given by} \]

$$WW(G) = \frac{1}{2}\{2n^2 + 12Pn - 25n - 2n - 10P - 6PS + 12P^2\}$$

Where

- n - cardinality of complete graph and n is even
- S - Number of Pendent vertices attached to alternative vertices of n.
- P - Total number of Pendent vertices present in G.

**Proof:** To find Hyper-Wiener index of the graph we need to find following two parts,

To find $W(G)$:

$$W(G) = \frac{1}{2}\sum_{(u,v)\in V(G)} d(u, v)$$

\[= \frac{1}{2}\left[ \left( (1 + 1 + \cdots + 1) + (2 + 2 + \cdots + 2) + \cdots + (1 + 1 + \cdots + 1) + (2 + 2 + \cdots + 2) \right) \right]
\]

\[n - 1 \text{ times} \quad P \text{ times} \quad n - 1 \text{ times} \quad P \text{ times} \]

\[+ \left[ (1 + 1 + \cdots + 1 + 2 + 2 + \cdots + 2) + \cdots + (1 + 1 + \cdots + 1 + 2 + 2 + \cdots + 2) \right] \]

\[n + S - 1 \text{ times} \quad P-S \text{ times} \quad n + S - 1 \text{ times} \quad P-S \text{ times} \]
\[ W(G) = \frac{1}{2} \left[ \left( n - 1 + 2P \right) + \frac{n}{2} \left[ n - 1 + S + 2P - 2S \right] + P \left[ 1 + 2(n + S - 2) + 3(P - s) \right] \right] \]

\[ W(G) = \frac{1}{4} \left[ 2n^2 + 8pn - 5n - 2n - 6P - 2PS + 6P^2 \right] \] \hspace{1cm} \text{(a)}

To find \( WW^*(G) \):

\[ WW^*(G) = \frac{1}{2} \sum_{(u,v) \in V(G)} d^2(u, v) \]

\[ WW^*(G) = \frac{1}{2} \left[ \left( 1 + 1 + \cdots + 1 \right) + \left( 1 + 1 + \cdots + 1 \right) + \cdots + \left( 1 + 1 + \cdots + 1 \right) \right] \]

\[ WW^*(G) = \frac{1}{4} \left[ 4Pn - Sn - 4SP + 6P^2 - 4P \right] \] \hspace{1cm} \text{(b)}

Since \( WW(G) = W(G) + WW^*(G) \)

Therefore
WW(G) = \frac{1}{4} \left( \frac{1}{4} \right) \left[ 2n^2 + 8Pn - 5n - 2n - 6P - 2PS + 6P^2 \right] + \frac{1}{4} \{ 4Pn - 5n - 4SP + 6P^2 - 4P \} \\

WW(G) = \frac{1}{4} \{ 2n^2 + 12Pn - 25n - 2n - 10P - 6PS + 12P^2 \}

**Corollary 2.2.1:** Let $K_n$ be the complete graph on $n$ vertices ($n$ is even). The graph $G$ obtained by attaching three pendant vertices to alternative vertices of $K_n (n \geq 4)$ with common vertex then its Hyper-Wiener index given by

WW(G) = \frac{1}{4} \{ 2n^2 + 12Pn - 8n - 28P + 12P^2 \}

**Proof:** Substituting $S=3$ in above theorem, gives the result.

**Corollary 2.2.2:** Let $K_n$ be the compete graph on $n$ vertices ($n$ is even). The graph $G$ obtained by attaching two pendant vertices to alternative vertices of $K_n (n \geq 4)$ with common vertex then its Hyper-Wiener index given by

WW(G) = \frac{1}{4} \{ 2n^2 + 12Pn - 6n - 22P + 12P^2 \}

**Proof:** Substituting $S=2$ in above theorem, gives the result.

**Corollary 2.2.3:** Let $K_n$ be the compete graph on $n$ vertices ($n$ is even). The graph $G$ obtained by attaching one pendant vertex to alternative vertices of $K_n (n \geq 4)$ with common vertex then its Hyper-Wiener index given by

WW(G) = \frac{1}{4} \{ 2n^2 + 12Pn - 4n - 16P + 12P^2 \}

**Proof:** Substituting $S=1$ in above theorem, gives the result.

**Illustrations:**

\[ |K_4| = 4, S = 2, P = 4, W(G) = 50 \quad WW(G) = 76 \]

\[ |K_6| = 6, S = 1, P = 3, W(G) = 57 \quad WW(G) = 81 \]
REFERENCES: