ON RARE $$(\lambda_X M_X)^\ast$$ - SETS AND RARE $$(\lambda_X M_X)^\ast$$ - CONTINUITY IN GTMS - SPACES

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Abstract: Popa [10] introduced the useful concept of rare continuity. The purpose of this paper is to highlight the notions of rare sets and dense sets in Generalized Topology and minimal structure space. Further, we study their properties using the closure and interior operators. We apply this notion of sets to define a new class of rarely $$(\lambda_X m_X)^\ast$$ - continuous functions and investigate their characterizations.

Keywords: $$(\lambda_X m_X)^\ast$$ - set, $$(\lambda_X m_X)^\ast$$ - set, $$(\lambda_X m_X)^\ast$$ - rare set, $$(\lambda_X m_X)^\ast$$ - dense set, rarely$$^\ast$$ - continuous function.

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INTRODUCTION

In 1979, Popa [10] initiated the useful notion of rare continuity as a generalization of weak continuity. The class of rarely continuous functions was further investigated by Long and Herrington [8] and Jafari [5]. The authors [6] [7] introduced and investigated a new class of functions called rarely g-continuous functions. In 2002, A.Csaszar [1] has introduced the notion of generalized neighborhood systems and generalized topological spaces. He also studied the notions of continuous functions and associated interior and closure operator. W.K.Min [17] also studied almost continuity in generalized topological spaces.

Popa and Noiri [11] initiated the concept of minimal structure which is a generalization of a topology on a given non-empty set. They also introduced the notion of $$m_X$$ - open sets and $$m_X$$ - closed sets and characterized those sets using $$m_X$$ - closure and $$m_X$$ - interior operators. In [15], the concept of generalized topology and minimal structures was introduced and some properties of closed sets were also investigated.

The purpose of this present paper is to introduce the concept of rare sets and dense sets in generalized topological spaces and minimal structures and investigate their properties. Here, we consider a strong space which consists of a set X, generalized topology on X and minimal structure on X. We call the space with generalized topology and minimal structure as (briefly GTMS space). The notion of rarely $$(\lambda_X m_X)^\ast$$ - continuous functions is also defined and their characteristics are obtained.
PRELIMINARIES

Let X be a non-empty set and μ be a collection of subsets of X. Then μ is called a generalized topology (briefly GT) on X iff \( \emptyset \in \mu \) and \( G_i \in \mu \) for \( i \in I \neq \emptyset \) implies \( G = \bigcup_{i \in I} G_i \in \mu \). We call the pair \( (X, \mu) \), a generalized topological space (briefly GTS) on X. The elements of \( \mu \) are called \( \mu \)-open sets and the complements are called \( \mu \)-closed sets. The generalized closure of a subset S of X, denoted by \( c_\mu(S) \), is the intersection of generalized closed sets including S and the interior of S, denoted by \( i_\mu(S) \), is the union of \( \mu \)-open sets contained in S.

Let \( \mu \) be a generalized topology on a non-empty set X and \( S \subseteq X \). The \( \mu \)-\( \alpha \)-closure (resp. \( \mu \)-semi closure, \( \mu \)-pre closure , \( \mu \)-\( \beta \) closure) of a subset S of X, denoted by \( c_\mu(S) \) [resp. \( c_\alpha(S), c_\sigma(S), c_\tau(S) \)] is the intersection of \( \mu \)-\( \alpha \)-closed (resp. \( \mu \)-semi closed, \( \mu \)-pre closed , \( \mu \)-\( \beta \) closed) sets including S. The \( \mu \)-\( \alpha \)-interior (resp. \( \mu \)-semi interior, \( \mu \)-pre interior , \( \mu \)-\( \beta \) interior) of a subset S of X denoted by \( i_\mu(S) \) (resp. \( i_\alpha(S), i_\sigma(S), i_\tau(S) \)) is the union of \( \mu \)-open [resp. \( \mu \)-semi open, \( \mu \)-pre open, \( \mu \)-\( \beta \) open] sets contained in S.

A space \((X, \lambda)\) is said to be quasi-topological space [3], if is closed under finite intersection.

Theorem 2.1[1] Let \((X, \lambda)\) be a generalized topological space. Then
1. \( c_\lambda(A) = X - i_\lambda(X \setminus A) \);
2. \( i_\lambda(A) = X - c_\lambda(X \setminus A) \);

Definition 2.2[16] Let \((X, \lambda)\) be a GTS and \( A \subseteq X \), then
(i) \( x \in \lambda \)-Int(A) if and only if there exists \( V \in \lambda \) such that \( x \in V \subseteq A \);
(ii) \( x \in \lambda \)-Cl(A) if and only if \( V \cap A \neq \emptyset \) for every \( \lambda \)-open set V containing x.

Definition 2.3[16] Let \((X, \lambda)\) be a GTS. For subsets A and B, the following properties hold:
(i) \( \lambda \)-Cl(\( X \setminus A \)) = \( X - \lambda \)-Int(A) and \( \lambda \)-Int(\( X \setminus A \)) = \( X - \lambda \)-Cl(A);
(ii) If \( X \setminus A \in \lambda \), then \( \lambda \)-Cl(A) = A and if \( A \in \lambda \), then \( \lambda \)-Int(A) = A;
(iii) If \( A \subseteq B \), then \( \lambda \)-Cl(A) \( \subseteq \) \( \lambda \)-Cl(B) and \( \lambda \)-Int(A) \( \subseteq \) \( \lambda \)-Int(B);
(iv) \( A \subseteq \lambda \)-Cl(A) and \( \lambda \)-Int(A) \( \subseteq \) A;
(v) \( \lambda \)-Cl(\( \lambda \)-Cl(A)) = \( \lambda \)-Cl(A) and \( \lambda \)-Int(\( \lambda \)-Int(A)) = \( \lambda \)-Int(A).

Definition 2.4[10] A rare set is a set S such that \( \text{Int}S = \emptyset \) and dense set is a set S such that \( \text{Cl}S = X \).

Definition 2.5[8] A function \( f : X \to Y \) is called a rarely continuous function if for each \( x \in X \) and each \( G \in O(Y, f(x)) \), there exist a rare set \( R_G \) with \( G \cap \text{cl}(R_G) = \emptyset \) and \( U \in O(X, x) \) such that \( f(U) \subseteq G \cup R_G \).

Definition 2.6[11] Let X be a non-empty set and \( P(X) \) the power set of X. A subfamily \( m_x \) of \( P(X) \) is called a minimal structure (briefly m-structure) on X if \( \emptyset \in m_x \) and \( X \in m_x \). Each member of \( m_x \) is said to be \( m_x \)-open and the complement of \( m_x \)-open set is said to be \( m_x \)-closed.
Definition 2.7[11]. Let X be a non-empty set and mₙ a minimal structure on X. For a subset A of X, mₙ - CIA and mₙ - IntA are defined as follows:

\[ mₙ - CIA = \cap \{ F : A \subseteq F, X \setminus F \in mₙ \} \]

\[ mₙ - IntA = \cup \{ H : H \subseteq A, H \in mₙ \} \]

Definition 2.8[11] Let (X, mₙ) be a minimal structure. For any subsets A, B of X the following hold:

(i) \( mₙ - Int(X \setminus A) = X \setminus mₙ - Cl(A) \).

(ii) \( A \in mₙ \Rightarrow mₙ - IntA = A \).

(iii) \( X \setminus A \in mₙ \Rightarrow mₙ - CIA = A \).

(iv) \( A \subseteq B, mₙ - Int(A) \subseteq mₙ - Int(B) \) and \( mₙ - Cl(A) \subseteq mₙ - Cl(B) \).

(v) \( mₙ - Int(mₙ - Int(A)) = mₙ - IntA \) and \( mₙ - Cl(mₙ - Cl(A)) = mₙ - CIA \).

Definition 2.10[10]. Let X be a non-empty set, which has a minimal structure mₙ and let A be a subset of X. The mₙ - frontier of A, denoted by mₙ - Fr(A), is defined by \( mₙ - Fr(A) = mₙ - Cl(A) \cap mₙ - Cl(X \setminus A) \).

Definition 2.11[10] Let X be a non-empty set and let \( \lambdaₙ \) be a generalized topology and mₙ a minimal structure on X. A triple (X, \( \lambdaₙ \), mₙ) is called a generalized topology and minimal structure space (briefly GTMS).

Let (X, \( \lambdaₙ \), mₙ) be a generalized topology and minimal structure space. A subset A of X is said to be \( \lambdaₙ \) - closed set if \( \lambdaₙ - Cl(mₙ - Cl(A)) = A \). The complement of \( \lambdaₙ \) - closed set is said to be \( \lambdaₙ \) - open.

Definition 2.12.[15] Let (X, \( \lambda \)) be a GTS. A subset A of a space (X, \( \lambda \)) is said to be \( \lambda \) - rare set if \( \text{cl}_{\lambda}(A) = \emptyset \).

Definition 2.13.[15] Let (X, \( \lambda \)) be a GTS. A subset A of a space (X, \( \lambda \)) is said to be \( \lambda \) - dense set if \( \text{cl}_{\lambda}(A) = X \).

3. On rare \( (\lambdaₙ \) mₙ\( )^* \) and dense \( (\lambdaₙ \) mₙ\( )^* \) sets

Definition 3.1. Let (X, \( \lambdaₙ \), mₙ) be a GTMS. A subset A of X is said to be an \( (\lambdaₙ \) mₙ\( )^* \) set if \( [\lambdaₙ - Int(mₙ - Int(A))] = A \) \( [\lambdaₙ - Cl(mₙ - Cl(A))] = A \).

Lemma 3.2. Let (X, \( \lambdaₙ \), mₙ) be a GTMS. For a subset A of X, the following hold

(i) \( (\lambdaₙ \) mₙ\( )^* - Cl(X - A) = X - (\lambdaₙ \) mₙ\( )^* - Int(A) \)

(ii) \( (\lambdaₙ \) mₙ\( )^* - Int(X - A) = X - (\lambdaₙ \) mₙ\( )^* - Cl(A) \)

Definition 3.3. Let (X, \( \lambdaₙ \), mₙ) be a GTMS. A subset A of a space X is said to be rare \( (\lambdaₙ \) mₙ\( )^* \) set if \( (\lambdaₙ \) mₙ\( )^* - Int(A) = \emptyset \).

Definition 3.4. Let (X, \( \lambdaₙ \), mₙ) be a GTMS. A subset A of a space X is said to be dense \( (\lambdaₙ \) mₙ\( )^* \) set if \( (\lambdaₙ \) mₙ\( )^* - Cl(A) = X \).
Theorem 3.5. Let \((X, \lambda_X, m_X)\) be a GTMS. Let \(A \subseteq X\), then \(A\) is a rare \((\lambda_X, m_X)^\ast\) set if \(A^c\) is a dense \((\lambda_X, m_X)^\ast\) set.

Proof: Obvious.

Theorem 3.6. A subset \(A\) of \(X\) is both an \((\lambda_X, m_X)^\ast\) set and a rare \((\lambda_X, m_X)^\ast\) set if it is the null set.

Proof: Let \(A\) be a \((\lambda_X, m_X)^\ast\) set. then \((\lambda_X, m_X)^\ast - \text{Int}(A) = A\), and a rare \((\lambda_X, m_X)^\ast\) set (i.e) \((\lambda_X, m_X)^\ast - \text{Int}(A) = \phi\). Then \((\lambda_X, m_X)^\ast - \text{Int}(A) = A = \phi\). (i.e) \(A\) is a null set.

Remark 3.7. A subset \(A\) of \(X\) can be both a rare \((\lambda_X, m_X)^\ast\) set and a dense \((\lambda_X, m_X)^\ast\) set which is shown in the following example.

Example 3.8. Let \(X = \{a, b, c\}\) be a GTMS, the arbitrary intersections of rare \((\lambda_X, m_X)^\ast\) sets need not be rare \((\lambda_X, m_X)^\ast\) sets, which is shown in the following example.

Theorem 3.9. A subset \(A\) of \(X\) is both a rare \((\lambda_X, m_X)^\ast\) set and a dense \((\lambda_X, m_X)^\ast\) set if there exists neither an \((\lambda_X, m_X)^\ast\) - open set contained in \(A\) nor an \((\lambda_X, m_X)^\ast\) - closed set containing \(A\), except for \(\emptyset\) and \(X\).

Proof: Obvious.

Remark 3.10. A subset \(A\) of \(X\) can be both a rare \((\lambda_X, m_X)^\ast\) set and a dense \((\lambda_X, m_X)^\ast\) set then \(A\) is neither an \((\lambda_X, m_X)^\ast\) - open set nor an \((\lambda_X, m_X)^\ast\) - closed set.

Theorem 3.11. Let \((X, \lambda_X, m_X)\) be a GTMS, the arbitrary intersections of rare \((\lambda_X, m_X)^\ast\) sets are rare \((\lambda_X, m_X)^\ast\) sets.

Remark 3.12. Finite union of rare \((\lambda_X, m_X)^\ast\) sets need not be rare \((\lambda_X, m_X)^\ast\) sets, which is shown in the following example.

Example 3.13. Let \(X = \{a, b, c, d\}\) be a GTMS, the arbitrary intersections of rare \((\lambda_X, m_X)^\ast\) sets need not be rare \((\lambda_X, m_X)^\ast\) sets, which is shown in the following example.

Theorem 3.14. A subset \(A\) of \(X\) is a dense [resp. rare] \((\lambda_X, m_X)^\ast\) set iff for every open [resp. closed] \((\lambda_X, m_X)^\ast\) set \(G\) satisfying \(A \subseteq G\) [resp. \(A \supseteq G\)] we have \((\lambda_X, m_X)^\ast \text{Cl}(A) \supseteq G\) [resp., \((\lambda_X, m_X)^\ast \text{Int}(A) \subseteq G\)].

Proof: Suppose that \(A\) is a dense \((\lambda_X, m_X)^\ast\) set. Let \(G\) be a \((\lambda_X, m_X)^\ast\) set with \(A \subseteq G\). Then \((\lambda_X, m_X)^\ast \text{Cl}(A) = X \supseteq G\). Conversely, let \(G = X\). Then \(G\) is a \((\lambda_X, m_X)^\ast\) set and \(A \subseteq G\), so \((\lambda_X, m_X)^\ast \text{Cl}(A) \supseteq G = X\), (i.e) \((\lambda_X, m_X)^\ast \text{Cl}(A) = X\). So, \(A\) is a dense \((\lambda_X, m_X)^\ast\) set.

Remark 3.15. A subset \(A\) of \(X\) is a rare \((\lambda_X, m_X)^\ast\) set if there exist no non-null \((\lambda_X, m_X)^\ast\) set contained in \(A\).

Theorem 3.16. \((\lambda_X, m_X)^\ast \text{Cl}(A)\) [resp., \((\lambda_X, m_X)^\ast - \text{Int}A\)] is a dense [resp., rare] \((\lambda_X, m_X)^\ast\) set whenever \(A\) is a dense [resp., rare] \((\lambda_X, m_X)^\ast\) set.

Proof: Obvious
Definition 3.17. Let \((X, \lambda_X, m_X)\) be a GTS. A subset \(A\) of a space \(X\) is said to be closed rare [resp., open dense] \((\lambda_X m_X)^r\) set if the set \(A\) is both a \((\lambda_X m_X)^r\) set and a rare \((\lambda_X m_X)^r\) set [resp., a \((\lambda_X m_X)^r\) set and a dense \((\lambda_X m_X)^r\) set].

Theorem 3.18. A subset \(A\) of a space \(X\) is a closed [resp., open] rare [resp., dense] \((\lambda_X m_X)^r\) set iff \(A\) is a \((\lambda_X m_X)^r\) set which does not contain any non-null \((\lambda_X m_X)^r\) set.

Proof: Obvious.

Definition 3.19. Let \(X\) be a non-empty set and \(A\) be a subset of \(X\). The \((\lambda_X m_X)^r\) - frontier of \(A\), denoted by \((\lambda_X m_X)^r\) - Fr(A) is defined by

\[
(\lambda_X m_X)^r\ - Fr(A) = (\lambda_X m_X)^r\ - Cl(A) \cap (\lambda_X m_X)^r\ - Cl(X \setminus A).
\]

Theorem 3.20. If a subset \(A\) of \(X\) is a dense \((\lambda_X m_X)^r\) set, then

\[
(\lambda_X m_X)^r\ - Fr(A) = X \setminus (\lambda_X m_X)^r\ - Int(A).
\]

Proof: Let \(A\) be a dense \((\lambda_X m_X)^r\) set, i.e., \((\lambda_X m_X)^r\) - Cl(A) = X, then

\[
(\lambda_X m_X)^r\ - Fr(A) = (\lambda_X m_X)^r\ - Cl(A) \cap (\lambda_X m_X)^r\ - Cl(X \setminus A)
\]

\[
= X \cap (\lambda_X m_X)^r\ - Cl(X \setminus A)
\]

\[
= (\lambda_X m_X)^r\ - Cl(X \setminus A)
\]

\[
= X \setminus (\lambda_X m_X)^r\ - Int(A).
\]

Theorem 3.21. If a subset \(A\) of \(X\) is both dense \((\lambda_X m_X)^r\) set and a rare \((\lambda_X m_X)^r\) set iff

\[
(\lambda_X m_X)^r\ - Fr(A) = X.
\]

Proof. Let \(A\) be a dense \((\lambda_X m_X)^r\) set and a rare \((\lambda_X m_X)^r\) set, then

\[
(\lambda_X m_X)^r\ - Fr(A) = (\lambda_X m_X)^r\ - Cl(A) \cap (\lambda_X m_X)^r\ - Cl(X \setminus A)
\]

\[
= X \cap (\lambda_X m_X)^r\ - Cl(X \setminus A)
\]

\[
= (\lambda_X m_X)^r\ - Cl(X \setminus A)
\]

\[
= X / (\lambda_X m_X)^r\ - Int(A)
\]

\[
= X.
\]

Conversely, Let \((\lambda_X m_X)^r\) - Fr(A) = X. Then \((\lambda_X m_X)^r\) - Cl(A) \cap Cl(X \setminus A) = X (i.e.)

\[
(\lambda_X m_X)^r\ - Cl(A) = X \text{ and } (\lambda_X m_X)^r\ - Cl(X \setminus A) = X.
\]

Therefore \((\lambda_X m_X)^r\) - Cl(A) = X implies \(A\) is a dense \((\lambda_X m_X)^r\) set and \((\lambda_X m_X)^r\) - Cl(X / A) = X / (\lambda_X m_X)^r\ - Int(A) = X, i.e.,

\[
(\lambda_X m_X)^r\ - Int A = \phi, \text{ so } A \text{ is a rare } (\lambda_X m_X)^r\text{ set.}
\]

Remark 3.22.

1. \(\phi\) is a closed rare \((\lambda_X m_X)^r\) set.
2. \(X\) is not a closed rare \((\lambda_X m_X)^r\) set, since \(X\) is a \((\lambda_X m_X)^r\) set but not a rare \((\lambda_X m_X)^r\) set.
3. Arbitrary union of closed rare \((\lambda_X m_X)^r\) set need not be closed rare \((\lambda_X m_X)^r\) sets.
4. Arbitrary intersection of closed rare \((\lambda_X m_X)^r\) sets are closed rare \((\lambda_X m_X)^r\) sets.

Theorem 3.23. Let \(g: (X, \lambda_X, m_X) \rightarrow (Y, \lambda_Y, m_Y)\) be a \((\lambda_X m_X^- \lambda_Y m_Y)^r\) - continuous and injective. Then \(g\) preserves rare
(\lambda_X m_X)^* set.

**Proof.** Suppose that \( R_G \) is a rare \((\lambda_X m_X)^* \) set but that \( g(R_G) \) is not a rare \((\lambda_Y m_Y)^* \) set. Then \((\lambda_Y m_Y)^* \text{Int}(R_G) \neq \emptyset \), so there exist a non-null \((\lambda_Y m_Y)^* \) contained in \( g(R_G) \), (i.e) \( V \subseteq g(R_G) \). Since \( g \) is injective, \( g^{-1}(V) \subseteq R_G \). Since \( g \) is \((\lambda_X m_X - \lambda_Y m_Y)^* \) continuous, \( g^{-1}(V) \) is a \((\lambda_X m_X)^* \) set. But this contradicts the fact that \( R_G \) is rare. So, \( g \) preserves rare \((\lambda_X m_X)^* \) set.

4. Rarely \((\lambda_X m_X - \lambda_Y m_Y)^* \) - continuous functions:

**Definition 4.1.** A function \( f: (X, \lambda_X, m_X) \to (Y, \lambda_Y, m_Y) \) is called rarely continuous \((\lambda_X m_X - \lambda_Y m_Y)^* \) if for each \((\lambda_Y m_Y)^* \) set \( G \) containing \( f(x) \), there exist a rare \((\lambda_Y m_Y)^* \) set \( R_G \) with \( G \cap (\lambda_Y m_Y)^* - \text{Cl}(R_G) = \emptyset \) and a \((\lambda_X m_X)^* \) set \( U \) containing \( x \) such that \( f(U) \subseteq G \cup R_G \).

**Theorem 4.2.** Let \( f: (X, \lambda_X, m_X) \to (Y, \lambda_Y, m_Y) \) be a function, then the following statements are equivalent.

(i) The function \( f \) is rarely \((\lambda_X m_X - \lambda_Y m_Y)^* \) continuous at \( x \in X \).

(ii) For each \((\lambda_Y m_Y)^* \) set \( G \) containing \( f(x) \), there exist a \((\lambda_X m_X)^* \) set \( U \) containing \( x \) such that \((\lambda_Y m_Y)^* - \text{Int} f(U) \cap \lambda_Y m_Y - \text{Int}(Y/G)] = \emptyset \).

(iii) For each \((\lambda_Y m_Y)^* \) set \( G \) containing \( f(x) \), there exist a \((\lambda_X m_X)^* \) set \( U \) containing \( x \) such that \((\lambda_Y m_Y)^* - \text{Int} f(U) \subseteq (\lambda_Y m_Y)^* - \text{Cl}(G) \cup R_G \).

**Proof:** (i) \( \Rightarrow \) (ii):

Let \( G \) be an \((\lambda_Y m_Y)^* \) set containing \( f(x) \). Since \( G \) is a \((\lambda_Y m_Y)^* \) set, \( G = (\lambda_Y m_Y)^* \text{Int}(G) \leq (\lambda_Y m_Y)^* \text{Int}((\lambda_Y m_Y)^* - \text{Cl}(G)) \). Also, \((\lambda_Y m_Y)^* \text{Int}((\lambda_Y m_Y)^* - \text{Cl}(G)) \subseteq \lambda_Y m_Y \subseteq \lambda_Y m_Y \text{Int} G \subseteq \lambda_Y m_Y \text{Int} f(U) \cap \lambda_Y m_Y - \text{Int}(Y/G)] \). So, \( f(U) \subseteq G \cup R_G \).

We have \((\lambda_Y m_Y)^* - \text{Int}(\lambda_Y m_Y)^* - \text{Int}(\lambda_Y m_Y)^* - \text{Cl}(G) \cup R_G) \subseteq [Y \setminus (\lambda_Y m_Y)^* - \text{Cl}(G)] \cup [Y \setminus (\lambda_Y m_Y)^* - \text{Cl}(G)] \subseteq [(\lambda_Y m_Y)^* - \text{Int}((\lambda_Y m_Y)^* - \text{Cl}(G) \cup (\lambda_Y m_Y)^* - \text{Int} R_G) \subseteq (\lambda_Y m_Y)^* - \text{Cl}(G)] \cup [Y \setminus (\lambda_Y m_Y)^* - \text{Cl}(G)] = \emptyset \).

(ii) \( \Rightarrow \) (iii) \((\lambda_Y m_Y)^* - \text{Int} f(U) \cap (\lambda_Y m_Y)^* \subseteq [Y \setminus G] \neq \emptyset \), \((\lambda_Y m_Y)^* - \text{Int} f(U) \subseteq Y \setminus (\lambda_Y m_Y)^* - \text{Int} (Y \setminus G) = (\lambda_Y m_Y)^* - \text{Cl}(G) \).

(iii) \( \Rightarrow \) (i): Let there exist a \((\lambda_Y m_Y)^* \) set \( G \) containing \( f(x) \) with the properties given in (iii). Then there exist a \((\lambda_Y m_Y)^* \) set \( U \) containing \( x \) such that \((\lambda_Y m_Y)^* - \text{Int} f(U) \subseteq (\lambda_Y m_Y)^* - \text{Cl}(G) \).

\[
\begin{align*}
f(U) &= [f(U) \setminus (\lambda_Y m_Y)^* - \text{Int} f(U)] \cup (\lambda_Y m_Y)^* - \text{Int} f(U) \\
& \subseteq [f(U) \setminus (\lambda_Y m_Y)^* - \text{Int} f(U)] \cup (\lambda_Y m_Y)^* - \text{Cl}(G) \\
& = [f(U) \setminus (\lambda_Y m_Y)^* - \text{Int} f(U)] \cup G \cup ((\lambda_Y m_Y)^* - \text{Cl}(G) \setminus G). 
\end{align*}
\]

Let \( R_1 = [f(U) \setminus (\lambda_Y m_Y)^* - \text{Int} f(U)] \cap (Y \setminus G) \) and \( R_2 = (\lambda_Y m_Y)^* - \text{Cl}(G) \).
Then $R_G = R_1 \cup R_2$ is a rare set such that $(\lambda_X m_X^{-})^* - \text{Cl}(R_G \cap G = \phi$ and $f(U) \subseteq G \cup R_G$. Therefore, $f$ is a rarely $(\lambda_X m_X^{-} - \lambda_Y m_Y)^*$ continuous function.

**Theorem 4.3.** Let $f: (X, \lambda_X, m_X) \rightarrow (Y, \lambda_Y, m_Y)$ be a rarely $(\lambda_X m_X^{-} - \lambda_Y m_Y)^*$ continuous function, then the following statements hold.

(i) For each $(\lambda_Y m_Y)^*$ set $G$ containing $f(x)$, there exist a rare $(\lambda_Y m_Y)^*$ set $R_G$ with $G \cap (\lambda_Y m_Y)^* - \text{Cl}(R_G) = \phi$ such that $x \in (\lambda_X m_X)^* - \text{Int}(f^{-1}(G \cup R_G))$.

(ii) For each $(\lambda_Y m_Y)^*$ set $G$ containing $f(x)$, there exist a rare $(\lambda_Y m_Y)^*$ set $R_G$ with $(\lambda_Y m_Y)^* \text{Cl}(G \cap R_G) = \phi$ such that $x \in (\lambda_X m_X)^* - \text{Int}(f^{-1}((\lambda_Y m_Y)^* - \text{Cl}(G \cup R_G)))$.

**Proof:** (i) Suppose that $G$ is $(\lambda_Y m_Y)^*$ set $G$ containing $f(x)$. Then there exist a rare set $R_G$ with $G \cap (\lambda_Y m_Y)^* - \text{Cl}(R_G) = \phi$, and a $(\lambda_X m_X)^*$ set $U$ containing $x$ such that $f(U) \subseteq G \cup R_G$. It follows that $x \in U \subseteq (f^{-1}(G \cup R_G))$, which implies that $x \in (\lambda_X m_X)^* - \text{Int}(f^{-1}(G \cup R_G))$.

(ii) Suppose that $G$ is $(\lambda_Y m_Y)^*$ set $G$ containing $f(x)$. Then there exist a rare set $R_G$ with $G \cap (\lambda_Y m_Y)^* - \text{Cl}(R_G) = \phi$ such that $x \in (\lambda_X m_X)^* - \text{Int}(f^{-1}(G \cup R_G))$. Since $G \cap (\lambda_Y m_Y)^* - \text{Cl}(R_G) = \phi$, $R_G \subseteq (Y \cap (\lambda_Y m_Y)^* - \text{Cl}(G)) \cup ((\lambda_Y m_Y)^* - \text{Cl}(G))$. Let $R_1 = (R_G \cup (Y \cap (\lambda_Y m_Y)^* - \text{Cl}(G))$. It follows that $R_1$ is a rare $(\lambda_Y m_Y)^*$ set with $(\lambda_Y m_Y)^* - \text{Cl}(R_1) = \phi$. Therefore $x \in (\lambda_X m_X)^* - \text{Int}(f^{-1}(G \cup R_G)) \subseteq (\lambda_X m_X)^* - \text{Int}(f^{-1}((\lambda_Y m_Y)^* - \text{Cl}(G \cup R_G)))$.

**Theorem 4.4.** If $f$ is a rarely $(\lambda_X m_X^{-} - \lambda_Y m_Y)^*$ continuous function, then there exist a clopen set $G$ such that each $(\lambda_X m_X)^*$ set $U$ containing $x$, $f(U)$ is a dense $(\lambda_Y m_Y)^*$ set.

**Proof:** If $G$ is a clopen subset of $Y$ then $G \cup R_G$ is a dense $(\lambda_Y m_Y)^*$ set. Then, $(\lambda_Y m_Y)^* - \text{Cl}(f(U)) \subseteq (\lambda_Y m_Y)^* - \text{Cl}(G \cup R_G) = Y$. Hence $(\lambda_Y m_Y)^* - \text{Cl}(f(U)) = Y$. (i.e.) $f(U)$ is a dense $(\lambda_Y m_Y)^*$ set.

**Remark 4.5.** Let $f: (X, \lambda_X, m_X) \rightarrow (Y, \lambda_Y, m_Y)$ be a rarely $(\lambda_X m_X^{-} - \lambda_Y m_Y)^*$ continuous function. Then there exists a $(\lambda_Y m_Y)^*$ set $G$ containing $f(x)$ and a rare $(\lambda_Y m_Y)^*$ set $R_G$ such that $G \cap (\lambda_Y m_Y)^* - \text{Cl}(R_G)$ is also a rare $(\lambda_Y m_Y)^*$ set.

**Theorem 4.6.** Let $f: (X, \lambda_X, m_X) \rightarrow (Y, \lambda_Y, m_Y)$ be a rarely $(\lambda_X m_X^{-} - \lambda_Y m_Y)^*$ continuous function. Then the graph function $g: X \rightarrow X \times Y$, defined by $g(x) = (x, f(x))$ for every $x$ in $X$, is a rarely $(\lambda_X m_X^{-} - \lambda_X m_X \times \lambda_Y m_Y)^*$ continuous functions.

**Proof:** Suppose that $x \in X$ and $W$ is any $(\lambda_X m_X \times \lambda_Y m_Y)^*$ set containing $g(x)$. It follows that there exists a $(\lambda_X m_X)^*$ set $U$ and $(\lambda_Y m_Y)^*$ set $V$ such that $(x, f(x)) \in U \times V \subseteq W$. Since $f$ is rarely $(\lambda_X m_X^{-} - \lambda_X m_X \times \lambda_Y m_Y)^*$ continuous, from theorem 4.3 there exists an $(\lambda_X m_X)^*$ set $G$ containing $x$ such that $(\lambda_X m_X \times \lambda_Y m_Y)^* - \text{Int}(G) \subseteq (\lambda_Y m_Y)^* - \text{Cl}(V)$. Let $E = U \cap G$. It follows that $E$ is a $(\lambda_X m_X)^*$ set containing $x$ for which $(\lambda_X m_X \times \lambda_Y m_Y)^* - \text{Int}(g(E)) \subseteq (\lambda_X m_X \times \lambda_Y m_Y)^* - \text{Int}(U \times V) \subseteq (\lambda_X m_X \times \lambda_Y m_Y)^* - \text{Cl}(W)$. Therefore $g$ is a rarely $(\lambda_X m_X \times \lambda_Y m_Y)^*$ continuous functions.

**Theorem 4.7.** Let $f: (X, \lambda_X, m_X) \rightarrow (Y, \lambda_Y, m_Y)$ be a rarely $(\lambda_X m_X^{-} - \lambda_Y m_Y)^*$ continuous function and $g: (Y, \lambda_Y, m_Y) \rightarrow (Z, \lambda_Z, m_Z)$ a one-to-one $(\lambda_Y m_Y^{-} - \lambda_Z m_Z)^*$ continuous functions. Then $g \circ f: (X, \lambda_X, m_X) \rightarrow (Z, \lambda_Z, m_Z)$ is a rarely $(\lambda_X m_X^{-} - \lambda_Z m_Z)^*$ continuous function.
Proof: Suppose that $x \in X$ and $g \circ f(x) \in V$, where $V$ is a $(\lambda_Y m_Y)^*$-set in $Z$. By hypothesis, $g$ is rarely $(\lambda_Y m_Y - \lambda_Z m_Z)^*$-continuous, therefore there exists a $(\lambda_Y m_Y)^*$ set $G \subseteq Y$ containing $f(x)$ such that $g(G) \subseteq V$. Since $f$ is rarely $(\lambda_X m_X - \lambda_Y m_Y)^*$-continuous, there exists a rare $(\lambda_Y m_Y)^*$ set $R_G$ with $G \cap (\lambda_Y m_Y)^* \text{Cl}(R_G) = \emptyset$, and a $(\lambda_X m_X)^*$ set $U$ containing $x$ such that $f(U) \subseteq G \cup R_G$. It follows from theorem 3.23, that $g(R_G)$ is a rare $(\lambda_Z m_Z)^*$ set in $Z$. Since $R_G$ is a subset of $Y \setminus G$, and $g$ is injective, we have $(\lambda_Y m_Y)^* \text{Cl}(g(R_G)) \cap V = \emptyset$. This implies that $g \circ f(U) \subseteq V \cup (g(R_G))$.

REFERENCES

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