A COMMON FIXED POINT THEOREM IN FUZZY METRIC SPACE USING SEMI-COMPATIBLE MAPPINGS

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Abstract: In this paper, the concept of semi-compatibility and weak compatibility in Fuzzy metric space has been applied to prove a common fixed point theorem. We generalize the result of Arihant Jain et.al. [11], using rational inequality.

Keyword: Fuzzy metric space, common fixed point, compatible maps, semi-compatible maps and weak compatible maps.

1. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [22] in 1965. Since then this concept was used in topology and analysis. Many authors have extensively developed the theory of fuzzy sets and its application. Especially, Deng[4], Erceg[5], Kaleva and Seikkala[12], Kramosil and Michalek[13] have introduced the concept of fuzzy metric space in different ways. George and Veeramani[8] have modified the concept of fuzzy metric space introduced by Kramosil and Michalek[13] and have defined the Hausdroff topology on fuzzy metric spaces. It has been seen that the study of Kramosil and Michalek[13] of fuzzy metric space covered almost all the points in the way for developing this theory to the field of fixed point theorem, in particular for the study of contractive type maps. They have also shown that every metric induces a fuzzy metric. Singh [21] proved various fixed point theorems using the concepts of semi-compatibility, compatibility and implicit relations in Fuzzy metric space. Kumar and Pant [15] have given a common fixed point theorem for two pairs of compatible mapping satisfying expansion type condition in probabilistic Menger space. Recently, Arihant Jain et.al.[11] improved the result of Kumar and Pant [15] by dropping the condition of continuity of the mapping and using semi and weak compatibility of the mapping in place of compatibility.

In this paper we are generalizing the result of Arihant Jain et.al.[11] using rational inequality.

2. PRELIMINARIES

Definition 2.1.[18] A binary operation * : [0,1] [0,1] → [0,1] is called a continuous t-norm if [0,1],* is an abelian topological monoid with unit 1 such that a * b ≤ c * d whenever a ≤ c and b ≤ d for all a,b,c,d ∈ [0,1]. Example of t-norm are a * b=ab and a * a=min{a,b}.
Definition 2.2. [13] The 3-tuple $(X, M, *)$ is said to be a Fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous t-norm and $M$ is a Fuzzy set in $X^2 \times \{0, \infty\}$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$.

(FM-1) $M(x, y, 0) = 0$.

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$.

(FM-3) $M(x, y, t) = M(y, x, t)$.

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$.

(FM-5) $M(x, y, t) : [0, \infty) \rightarrow [0, 1]$ is left continuous.

Where $M(x, y, t)$ can be considered as the degree of nearness between $x$ and $y$ with respect to $t$. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$.

Definition 2.3. [9] Let $(X, M, *)$ be a fuzzy metric space:

1. A sequence $x_n$ in $X$ is said to be convergent to a point $x \in X$ (denoted by $\lim_{n \to \infty} x_n = x$), if $\lim_{n \to \infty} M(x_n, x, t) = 1$ for all $t > 0$.

2. A sequence $x_n$ in $X$ is said to be a Cauchy sequence if $\lim_{n \to \infty} M(x_n, x_{n+p}, t) = 1$ for all $t > 0$ and $p > 0$.

3. A Fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Let $(X, M, *)$ be a fuzzy metric space with following condition:

(FM-6) $\lim_{t \to 0} M(x, y, t) = 1$ for all $x, y \in X$.

Definition 2.4. [19] A function $M$ is continuous in Fuzzy metric space if and only if whenever $x_n \to x, y_n \to y$, then $\lim_{t \to 0} M(x_n, y_n, t) = M(x, y, t)$ for all $t > 0$.

Definition 2.5. [14] Let $A$ and $B$ be mappings from a Fuzzy metric space $(X, M, *)$ into itself. The mappings $A$ and $B$ are said to be weakly compatible if they commute at their coincidence points, i.e. $Ax = Bx$ implies $ABx = BAx$.

Definition 2.6. [21] Suppose $A$ and $S$ be two maps from a Fuzzy metric space $(X, M, *)$ into itself. Then they are said to be semi-compatible if $\lim_{n \to \infty} ASx_n = Sx$ whenever $x_n$ is a sequence such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \in X$. 

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In [9] Grabiec has given two important lemmas for contraction condition. We have the following lemmas for expansion type condition.

Lemma 2.1. Let \( \{x_n\} \) be a sequence in a Fuzzy metric space \( (X, M, *) \) with (FM-6). If there exists a number \( h > 1 \) such that
\[
M(x_{n+1}, x_n, h) \leq M(x_{n+2}, x_{n+1}, t)
\]
for all \( t > 0 \) and \( n = 1, 2, 3, \ldots \) Then \( x_n \) is Cauchy sequence in \( X \).

Lemma 2.2. If for all \( x, y \in X, t > 0 \) and for a number \( h > 1 \),
\[
M(x, y, h) \leq M(x, y, t)
\]
then \( x = y \)

3. MAIN RESULT

In this section we present our main result.

Theorem 3.1: Let \( (X, M, *) \) be a complete Fuzzy metric space where * is continuous \( t \)-norm and satisfies \( x * x \geq x \) for all \( x \in [0, 1] \). Let \( A, B, S \) and \( T \) be self mappings of a Fuzzy metric space satisfying the following conditions:

(3.1.1) \( A \) and \( B \) are surjective.
(3.1.2) \( (A, S) \) is semi-compatible and \( (B, T) \) is weakly compatible.
(3.1.3) \( u, v \in X \) and \( h > 1 \),
\[
M(v_{2n+1}, v_{2n+1}, t) \leq M(v_{2n+2}, v_{2n+2}, t)
\]
Where \( a, b, r, s \geq 0 \) with \( a \) & \( b \) and \( r \) & \( s \) cannot be simultaneously 0.

Proof: Let \( u_0 \in X \). Since \( A \) and \( B \) are surjective, we choose a point \( u_1 \in X \) such that
\[
Au_1 = Tu_0 = v_0
\]
and for this point \( u_1 \), there exists a point \( u_2 \in X \) such that
\[
Bu_2 = Su_1 = v_1.
\]
Continuing in this manner, we obtain a sequence \( \{v_n\} \) in \( X \) as follows:
\[
Au_{2n+1} = Tu_{2n} = v_{2n}
\]
and \( Bu_{2n+2} = Su_{2n+1} = v_{2n+1} \).

Using (3.1.3), we have
\[
M(v_{2n}, v_{2n+1}, t) = M(Au_{2n+1}, Bu_{2n+2}, h)
\]
\[
\leq \min \left\{ M(T_{2n+2}, Bu_{2n+2}, t), \frac{rM(v_{2n+1}, v_{2n+2}, t) + sM(v_{2n+1}, v_{2n+2}, t)}{r + sM(v_{2n+2}, v_{2n+2}, t)} \right\}
\]
\[
\leq \min \left\{ M(v_{2n+2}, v_{2n+2}, t), \frac{r + sM(v_{2n+1}, v_{2n+2}, t)}{r + sM(v_{2n+2}, v_{2n+2}, t)} \right\}
\]
\[
\leq \min \left\{ M(v_{2n+2}, v_{2n+2}, 1), 1 \right\}
\]
\[
\leq M(v_{2n+1}, v_{2n+1}, t)
\]
Therefore, by Lemma 2.1, \( \{v_n\} \) is a Cauchy sequence.

Since \( X \) is complete, \( \{v_n\} \) converges to some point \( z \in X \). Consequently, the subsequences \( \{Au_{2n+1}\}, \{Bu_{2n}\}, \{Su_{2n+1}\} \) and \( \{Tu_{2n}\} \) also converges to \( z \).

Since \( A \) and \( S \) are semi-compatible, so \( \lim_{n \to \infty} ASu_{2n+1} = Sz. \) Also, \( \lim_{n \to \infty} ASu_{2n+1} = Az. \)

Since the limit in Fuzzy metric space is unique, we get \( Az = Sz. \)

Again by (3.1.3),
\[
M(Az, Bu_{2n}, h) \leq \min \left\{ M(Tu_{2n}, Bu_{2n}, t), \frac{rM(Sz, Bu_{2n}, t) + sM(Sz, Tu_{2n}, t)}{r + sM(Bu_{2n}, Tu_{2n}, t)} \right\}
\]
Letting \( n \to \infty \), we get

\[
M(Az, z, h_t) \leq \min \left\{ M(z, z, t), \frac{rM(Sz, z, t) + sM(Sz, z, t)}{r + sM(Sz, z, t)} \right\},
\]

\[
\leq \min \{1, M(Sz, z, t)\},
\]

\[
\leq M(Sz, z, t).
\]

which gives \( M(Az, z, h_t) \leq M(Az, z, t) \) implying \( Az = z \). Therefore, \( Az = Sz = z \).

Now suppose \( z = Bp \) for some \( p \in X \), then by (3.1.3)

\[
M(Au_{2n+1}, Bp, h_t) \leq \min \left\{ M(Tp, Bp, t), \frac{rM(Su_{2n+1}, Bp, t) + sM(Su_{2n+1}, Tp, t)}{r + sM(Bp, Tp, t)} \right\},
\]

Letting \( n \to \infty \), we get

\[
M(z, z, h_t) \leq \min \left\{ M(Tp, z, t), \frac{rM(z, z, t) + sM(z, Tp, t)}{r + sM(z, Tp, t)} \right\},
\]

\[
\leq \min \{ M(Tp, z, t), 1 \},
\]

\[
\leq M(Tp, z, t).
\]

which gives \( M(z, z, h_t) \leq M(z, Tp, t) \). It gives that \( z = Tp \). Therefore, \( Bp = Tp = z \).

\( \Rightarrow p \) is a coincidence point of \( B \) and \( T \).

Since \( B \) and \( T \) are weakly compatible and \( Bp = Tp = z \).
Therefore, \( TBp = BTP \) implies \( Tz = Bz \).

Now by (3.1.3), and using \( Tz = Bz \),

\[
M(Au_{2n+1}, Bz, h_t) \leq \min \left\{ M(Tz, Bz, t), \frac{rM(Su_{2n+1}, Bz, t) + sM(Su_{2n+1}, Tz, t)}{r + sM(Bz, Tz, t)} \right\},
\]

Letting \( n \to \infty \), we get

\[
M(z, Tz, h_t) \leq \min \left\{ M(Tz, Tz, t), \frac{rM(Tz, Tz, t) + sM(Tz, Tz, t)}{r + sM(Tz, Tz, t)} \right\},
\]

\[
\leq \min \{1, M(Tz, Tz, t)\},
\]

\[
\leq M(Tz, Tz, t),
\]

which gives \( M(z, Tz, h_t) \leq M(z, Tz, t) \). It gives that \( z = Tz \). Hence, \( Bz = z \).

Therefore, \( Az = Bz = Tz = Sz = z \).

Hence, \( z \) is a common fixed point of \( A, B, S \) and \( T \).

Finally for the uniqueness of fixed point, let \( w \in X \) be another fixed point.

Then from (3.1.3),

\[
M(Aw, Bz, h_t) \leq \min \left\{ M(Tz, Bz, t), \frac{rM(Sw, Bz, t) + sM(Sw, Tz, t)}{r + sM(Bz, Tz, t)} \right\},
\]

\[
M(w, z, h_t) \leq \min \left\{ M(z, z, t), \frac{rM(w, z, t) + sM(w, z, t)}{r + sM(z, z, t)} \right\},
\]

\[
\leq \min \{1, M(w, z, t)\},
\]

\[
\leq M(w, z, t),
\]

Which gives \( M(w, z, h_t) \leq M(w, z, t) \), \( \Rightarrow w = z \).

This completes the proof of the theorem.

**Remark 3.1:** We generalized the result of Arihant Jain et.al.[11] using rational inequality.

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