TERMINAL ZAGREB ECCENTRICITY INDICES OF LINE GRAPHS

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Abstract: Terminal Zagreb eccentricity indices were proposed analogously to Zagreb eccentricity indices. For a connected graph, the first Terminal Zagreb eccentricity index is defined as the sum of the squares of the eccentricities of the terminal vertices, and the second Zagreb eccentricity index is defined as the sum of the products of the eccentricities of all pairs of terminal vertices. In this paper we obtain results for the terminal Zagreb eccentricity indices of line graphs.

KEYWORDS: Zagreb eccentricity index, Terminal Zagreb eccentricity index, Line graph.

1. INTRODUCTION

Let G be a connected graph with vertex set V (G) = {v1, v2, . . . , vn} and edge set E(G) = {e1, e2, . . . , en}. The degree of a vertex v in G is the number of edges incident to it and is denoted by d(v) or degG (v). If degree of v is one then v is called a pendant vertex or terminal vertex. An edge e = uv of a graph G is called a pendant edge if d(u) = 1 or d(v) = 1. The line graph of a connected graph G, denoted by L(G) is the graph whose vertices are the edges of G and two vertices of L(G) are adjacent whenever the corresponding edges of G are adjacent. The distance between the vertices vi and vj in G is equal to the length of a shortest path joining them and is denoted by d(vi, vj)G. For a vertex vi its eccentricity, ei is the largest distance from vi to any other vertices of G.

The first and the second Zagreb eccentricity indices [6, 9] are defined as follows,

\[ E_1 = E_1(G) = \sum_{v_i \in V(G)} e_i^2 \quad \text{...... (1.1)} \]

\[ E_2 = E_2(G) = \sum_{v_i \in V(G)} e_i e_j \quad \text{...... (1.2)} \]

Analogously to Zagreb eccentricity indices, defining the first and the second Terminal Zagreb eccentricity indices as,

\[ T[E_1(G)] = \sum_{v_i \in V_T(G)} e_i^2 \quad \text{...... (1.3)} \]

\[ T[E_2(G)] = \sum_{v_i, v_j \in V_T(G)} e_i e_j \quad \text{...... (1.4)} \]

Where, V_T (G) = {v1, v2, . . . , vl} is the set of all pendent vertices of G.

Defining the set D2(G) as,

\[ D_2(G) = \{v \mid \text{deg}_G(v) = 2 \text{ and one neighbour of } v \text{ is pendent} \}. \]

2. EXISTING RESULTS

Many researchers have studied and obtained several results on Zagreb eccentricity indices of various graphs [1, 2, 3, 4, 5, 7, 10]

Recently H. S. Ramane et. al. [8] have obtained expressions for terminal weiner index of Line graphs
Theorem 2.1 [8]: Let $G$ be a connected graph with $n \geq 4$ vertices and let $D_2(G) = \{v_1, v_2, \ldots, v_q\}$. Then

$$TW (L(G)) = \sum_{i \neq j \in \mathcal{E}} \deg(v_i, v_j / G) + \frac{q(q-1)}{2}$$

Corollary 2.2 [8]: $TW(L(G)) = 0$ if and only if the graph $G$ satisfies one of the following conditions. (i) $G$ has no edge $e = uv$ where $\deg_G(u) = 1$ and $\deg_G(v) = 2$. (ii) $G$ has only one edge $e = uv$ where $\deg_G(u) = 1$ and $\deg_G(v) = 2$. (iii) $G$ has no pendant vertices. (iv) $G$ has only one pendant vertex. (v) $G$ has no vertex of degree 2.

Theorem 2.3 [8]: Let $G$ be a connected graph with $n \geq 4$ vertices and $G'$ be the graph obtained from $G$ by removing pendant vertices of $G$. If $p$ is the number of pendant vertices of $G'$, then

$$TW(L(G)) \leq TW(G') + \frac{p(p-1)}{2}$$

Equality holds if and only if (i) $G = K_{1,n-1}$ or (ii) $G$ has no Bridge $e$ such that one of the components of $G-e$ is $K_{1,n}$, $s \geq 2$ and $G \neq K_{1,n-1}$.

Corollary 2.4 [8]: Let $G$ be a connected graph with $n \geq 4$ vertices and $G'$ be the graph obtained from $G$ by removing pendant vertices. Let $p$ be the number of pendant vertices of $G'$. If all pendant edges of $G$ are mutually independent, then

$$TW(L(G)) = TW(G') + \frac{p(p-1)}{2}$$

3. TERMINAL ZAGREB ECCENTRICITY INDICES OF LINE GRAPHS

Theorem 3.1: Let $G$ be a connected graph with $n \geq 4$ vertices and $D_2(G) = \{v_1, v_2, \ldots, v_q\}$. Then the terminal Zagreb first and second eccentricity index of line graph of $G$ is,

$$T[E_1(L(G))] = \sum_{v_i \in D_2(G)} e_i^2$$

$$T[E_2(L(G))] = \sum_{v_i, v_j \in D_2(G)} e_i e_j$$

Proof: Let $G$ be a connected graph with $n \geq 4$ vertices and $D_2(G) = \{v_1, v_2, \ldots, v_q\}$. Let $E_2 = \{e_1, e_2, \ldots, e_q\}$ be the set of pendant edges of $G$. We know that if $e_i = uv \in E_2$, where $E_2 \subseteq E_i$ then $\deg_G(u) = 1$ and $\deg_G(v) = 2$, $i = 1, 2, \ldots, q$.

Consider two edges $e_i$ and $e_j$ with $e_i = uv \in E_2$ and $e_j = vw \in E_2$.

Where, $\deg_G(u) = 1 = \deg_G(w)$ and $\deg_G(v) = \deg_G(v), i = 1, 2, \ldots, q$.

Therefore $e_i$ and $e_j$ are the pendant vertices of $L(G)$.

Therefore,

$$T[E_1(L(G))] = \sum_{v_i \in D_2(G)} e_i^2$$

$$T[E_2(L(G))] = \sum_{v_i, v_j \in D_2(G)} e_i e_j$$

Corollary 3.2: $T[E_1(L(G))] = T[E_2(L(G))] = 0$ if and only if the graph $G$ satisfies one of the following conditions. (i) $G$ has no edge $e = uv$ where $\deg_G(u) = 1$ and $\deg_G(v) = 2$. (ii) $G$ has only one edge $e = uv$ where $\deg_G(u) = 1$ and $\deg_G(v) = 2$. (iii) $G$ has only one pendant vertex. (iv) $G$ has no pendant vertex. (v) $G$ has no vertex of degree 2.

Theorem 3.3: Let $G$ be a connected graph with $n \geq 4$ vertices and $G'$ be the graph obtained from $G$ by removing pendant vertices of $G$. If $p$ is the number of pendant vertices of $G'$, then

$$T[E_1(L(G))] \leq \sum_{v_i \in D_2(G)} e_i^2 + \sum_{v_i \in V_1(G')} e_i^2$$

$$T[E_2(L(G))] \leq \sum_{v_i \in D_2(G)} e_i e_j + \sum_{v_i \in V_1(G')} e_i e_j$$

Equality holds if and only if (i) $G = K_{1,n-1}$ or (ii) $G$ has no Bridge $e$ such that one of the components of $G-e$ is $K_{1,n}$, $s \geq 2$ and $G \neq K_{1,n-1}$.

Proof: Let $D_2(G) = \{v_1, v_2, v_3, \ldots, v_q\}$. The number of pendant vertices of $G'$ is at least $q$. If $p$ is the number of pendant vertices in $G'$ then $p \geq q$ from Theorem 3.1.

$$T[E_1(L(G))] = \sum_{v_i \in D_2(G)} e_i^2 \leq \sum_{v_i \in D_2(G)} e_i^2 + \sum_{v_i \in V_1(G')} e_i^2$$

$$T[E_2(L(G))] = \sum_{v_i \in D_2(G)} e_i e_j \leq \sum_{v_i \in D_2(G)} e_i e_j + \sum_{v_i \in V_1(G')} e_i e_j$$

Therefore,

$$T[E_1(L(G))] \leq \sum_{v_i \in D_2(G)} e_i^2 + \sum_{v_i \in V_1(G')} e_i^2$$

$$T[E_2(L(G))] \leq \sum_{v_i \in D_2(G)} e_i e_j + \sum_{v_i \in V_1(G')} e_i e_j$$

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For equality we consider the following cases:

Case I: for \( G = K_{1,n} \), obviously equality holds.

Case II: If \( G \neq K_{1,n} \) and if there is an edge in \( G \) such that one of the component of \( G - e \) is \( K_{1,t} \), \( s \geq 2 \) then \( q = p \). Therefore

\[
\sum_{v \in e_{G}(G)} e_i^2 = \sum_{v \in e_{G}(G')} e_i^2 \quad \text{and} \quad \sum_{(v_i, v_j) \in e_{G}(G)} e_j = \sum_{(v_i, v_j) \in e_{G}(G')} e_j
\]

i.e., \( |E_1(L(G))] = |E_1(L(G'))| \) and \( |E_2(L(G))| = |E_2(L(G'))| \)

Conversely, let \( G \) contains a bridge \( e \) such that one of the component of \( G - e \) is \( K_{1,t} \), \( s \geq 2 \) Therefore \( p > q \).

\[
\sum_{v \in e_{G}(G)} e_i^2 < \sum_{v \in e_{G}(G')} e_i^2
\]

From Theorem 3.1

\[
|E_1(L(G))] = \sum_{v \in e_{G}(G)} e_i^2 < \sum_{v \in e_{G}(G')} e_i^2 \quad \text{and} \quad |E_2(L(G))| = \sum_{(v_i, v_j) \in e_{G}(G)} e_j < \sum_{(v_i, v_j) \in e_{G}(G')} e_j
\]

By Eq. (3.1), \( q < p \)

\[|E_1(L(G))] < |E_1(L(G'))| \quad \text{and} \quad |E_2(L(G))] < |E_2(L(G'))|\]

Which is a contradiction.

This completes the proof.

**Corollary 3.4:** Let \( G \) be a connected graph with \( n \geq 4 \) vertices and \( G' \) be the graph obtained from \( G \) by removing pendent vertices of \( G \). If \( p \) is the number of pendent vertices of \( G' \). If all pendent edges of \( G \) are mutually independent, then \( |E_1(L(G))] = |E_1(L(G'))| \) and \( |E_2(L(G))] = |E_2(L(G'))| \)

**Proof:** Follows from the equality part of Theorem 3.3.

4. Terminal Zagreb eccentricity index of line graphs of some graphs

Let the vertices of \( G \) be \( v_1, v_2, \ldots, v_n \), then \( G^* \) is the graph obtained from \( G \) by adding \( n \) new vertices \( v'_1, v'_2, \ldots, v'_n \) and joining \( v'_i \) to \( v_i \) an edge, \( i = 1, 2, \ldots, n \).

**Theorem 4.1:** Let \( G \) be a connected graph with \( k \) pendent vertices, then

\[T[E_1(L(G^*))] = \sum_{e \in D(G^*)} e^2 \quad \text{and} \quad T[E_2(L(G))] = \sum_{(v_i, v_j) \in e_{G}(G)} e_j.
\]

**Proof:** If \( G \) has \( n \) vertices of which \( k \) are pendent vertices, then \( G^* \) has \( n \) pendent edges of which \( k \) pendent edges are such that for each \( e_i = uv \), \( i = 1, 2, \ldots, k \).

\[\text{deg}_{G^*}(u) = 1 \quad \text{and} \quad \text{deg}_{G^*}(v) = 2\]

Therefore \( \text{deg}_{L(G^*)}(u) = \text{deg}_{G^*}(u) + \text{deg}_{G^*}(v) - 2 = 1 \).

\( L(G^*) \) has \( k \) pendent vertices.

We know that pendent vertices of \( G^* \) are mutually independent, from Corollary 3.4,

\[T[E_1(L(G^*))] = \sum_{e \in D(G^*)} e^2 \quad \text{and} \quad T[E_2(L(G))] = \sum_{(v_i, v_j) \in e_{G}(G)} e_j.
\]

**Theorem 4.2:** \( T[E_1(L(S^+_n))] = T[E_1(K^*_{n-1})] \) and \( T[E_2(L(S^+_n))] = T[E_2(K^*_{n-1})] \).

**Proof:** Star graph \( S_n \) has \( n - 1 \) pendent vertices \( S^+_n \) has \( n \) pendent vertices and \( L(S^+_n) \) has \( n - 1 \) pendent vertices. Therefore terminal Zagreb eccentric indices of \( L(S^+_n) \) is same as the terminal Zagreb eccentric indices of \( K^*_{n-1} \).

Therefore, the result follows.

**Theorem 4.3:** Let \( G \) be a connected graph with \( k \) pendent vertices and \( H_t = L(H^*_{t-1}) \), \( t = 1, 2, \ldots \) Where \( H_0 = G \) and \( H_1 = L(G^*) \) then,

\[T[E_1(H_t)] = \sum_{i=1}^{k} (e_i + t)^2 \quad \text{and} \quad T[E_2(H_t)] = \sum_{i=1}^{k} (e_i + t)(e_i + t).
\]

**Proof:** As \( G \) has \( k \) pendent vertices from Theorem 4.1, the graph \( H_t \) has \( k \) pendent vertices, \( t = 1, 2, \ldots \)
Therefore, \( T[E_i(H_j)] = T[E_i(L(G^+))] = \sum_{i=1}^{k} (e_i + 1)^2 \).

By induction, let \( T[E_i(H_{t+1})] = T[E_i(L(H_{t+1}^+))] = \sum_{i=1}^{k} (e_i + t - 1)^2 \).

Therefore \( T[E_i(H_j)] = T[E_i(L(H_{t+1}^+))] = \sum_{i=1}^{k} |e_i + (t - 1) - 1|^2 \).

Hence, \( T[E_i(H_j)] = \sum_{i=1}^{k} (e_i + t)^2 \).

Similarly, \( T[E_2(H_j)] = \sum_{i<j} (e_i + t)(e_j + t) \).

\( \Box \)

References:


