A NOTE ON NEW DEFINITION OF FUZZY COMPACT SPACE ON THE BASIS OF REFERENCE FUNCTION

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Abstract: It is believed that the union of a fuzzy set and its complement may not be equal to the whole universal set. But it is seen that when we discussed fuzzy set on the basis of reference function then union of fuzzy set and its complement is equal to the whole universal set. In this article we will try to give definition of fuzzy compact on the basis of reference function.

Key words Fuzzy membership function, fuzzy reference function, fuzzy membership value, fuzzy open cover, fuzzy compact.

INTRODUCTION

Fuzzy set theory was discovered by Professor Zadeh [1] in 1965. Chang [2] introduced fuzzy topology. After the introduction of fuzzy sets and fuzzy topology, several researches were conducted on the generalizations of the notions of fuzzy sets and fuzzy topology. The concept of intuitionistic fuzzy sets was introduced by Atanassov [3] as a generalization of fuzzy sets. In 1997 Coker [4] introduced the concept of intuitionistic fuzzy topological spaces. The theory of fuzzy sets actually have been a generalization of the classical theory of sets in the sense that the theory of sets should have been a special case of the theory of fuzzy sets. But unfortunately it has been accepted that for fuzzy set A and its complement $A^C$, neither $A\cap A^C$ is empty set nor $A\cup A^C$ is the universal set. Whereas the operations of union and intersection of crisp sets are indeed special cases of the corresponding operation of two fuzzy sets, they end up giving peculiar results while defining $A\cap A^C$ and $A\cup A^C$.

Now if we consider first the usual definition of fuzzy sets. From figure-1, let A be fuzzy sets and is defined as $A=\{x, \mu(x), x \in X\}$. Its complement $A^C$, is characterized by (from fig-1) $A^C=\{x, (1-\mu(x)); x \in X\}$. Now from figure-1 it is clear that A and $A^C$ have something common, that is why it has been accepted that $A\cap A^C\neq \phi$, also from figure-1 it is clear that $A\cup A^C \neq X$. For these two inequalities, it has been accepted that the fuzzy sets do not form a field.
Not everything can be counted from zero level. For example, say at a particular place a mineral is available from a depth of 50 meters to a depth of 200 meters. Indeed, at that place the actual depth of the mineral bed is 150 meters, but it is to be counted from the depth of 50 meters. In other words, the depth of a mineral bed cannot be counted from the zero level which in this case is the surface of the earth. We start with this simple reasoning to extend the definition of fuzzy sets.

Consider an upside down view of the aforesaid mineral bed (given in the figure-2). Say for a given x measured from some point of reference on the surface of the earth, the mineral is available up to a depth or height in the upside down view $h_1(x)$ to a depth, or height in the upside down view, $h_2(x)$ so that thickness of the mineral bed is $(h_2(x)-h_1(x))$. One can see that $h_2(x)$ is like a membership function measured from a function of reference $h_1(x)$ such that $(h_2(x)-h_1(x))$ can be said to be the actual value of membership.

Baruah [5, 6] gave an extended definition of complementation of fuzzy sets. According to Baruah [5, 6] to define a fuzzy set, two functions namely fuzzy membership function and fuzzy reference function are necessary. Fuzzy membership value is the difference between fuzzy membership function and fuzzy reference function.

Let $\mu_1(x)$ and $\mu_2(x)$ be two functions such that $0 \leq \mu_2(x) \leq \mu_1(x) \leq 1$. For fuzzy number denoted by $\{x, \mu_1(x), \mu_2(x); x \in U\}$, we call $\mu_1(x)$ as fuzzy membership function and $\mu_2(x)$ a reference function such that $(\mu_1(x) - \mu_2(x))$ is the fuzzy membership value.

APPLICATION OF THE EXTENDED DEFINITION

Let $A = \{x, \mu_1(x), \mu_2(x); x \in U\}$ and $B = \{x, \mu_3(x), \mu_4(x); x \in U\}$ be two fuzzy sets.

Then we would have according to our way

$A \cap B = \{x, \min (\mu_1(x), \mu_3(x)), \max (\mu_2(x), \mu_4(x)); x \in U\}$ and $A \cup B = \{x, \max (\mu_1(x), \mu_3(x)), \min (\mu_2(x), \mu_4(x)); x \in U\}$.

Two fuzzy sets $C = \{x, \mu_c(x); x \in U\}$ and $D = \{x, \mu_d(x); x \in U\}$ in the usual definition would be expressed as $C = \{x, \mu_c(x), 0; x \in U\}$ and $D = \{x, \mu_d(x), 0; x \in U\}$.
Now for two fuzzy sets \( A = \{ x, \mu(x), 0 ; x \in X \} \) and 
\( B = \{ x, 1, \mu(x); x \in X \} \) are defined over the same universe \( X \). Then we would have 
\[
A \cap B = \{ x, \min (\mu(x), 1), \max (\mu(x), \mu(x)); x \in U \}
\]
which is nothing but the null set. In other words, \( B \) defined above is nothing but \( A^c \) complement in the classical sense of the set theory. That is if we define fuzzy sets \( A^c = \{ x, 1, \mu(x); x \in X \} \), it can be seen that it is nothing but the complement of the fuzzy sets 
\( A = \{ x, \mu(x), 0 ; x \in X \} \).

Also we have 
\[
A \cup B = \{ x, \max (\mu(x), 1), \min (0, \mu(x)); x \in U \}
\]
\[
= \{ x, 1, 0; x \in U \},
\]
which is nothing but the universal set.

We therefore conclude that if we express the complement of a fuzzy set \( A = \{ x, \mu(x), 0; x \in X \} \) as 
\( A^c = \{ x, 1, \mu(x); x \in X \} \), then we get
1. \( A \cap A^c = \phi \), and
2. \( A \cup A^c = \mathbb{U} \).

This would enable us to establish that the fuzzy sets do form a field if we define complementation in our way.

**PRELIMINARY RESULTS**

I. Basic operations

Let \( A = \{ x, \mu_1(x), \mu_2(x); x \in U \} \) and \( B = \{ x, \mu_3(x), \mu_4(x); x \in U \} \) be two fuzzy sets defined over the same universe \( U \).

1. \( A \subseteq B \) iff \( \mu_1(x) \leq \mu_3(x) \) and \( \mu_4(x) \leq \mu_2(x) \) for all \( x \in U \).
2. \( A \cup B = \{ x, \max (\mu_1(x), \mu_3(x)), \min (\mu_2(x), \mu_4(x)) \} \) for all \( x \in U \).
3. \( A \cap B = \{ x, \min (\mu_1(x), \mu_3(x)), \max (\mu_2(x), \mu_4(x)) \} \) for all \( x \in U \).

If for some \( x \in U \), \( \min (\mu_1(x), \mu_2(x)) \leq \max (\mu_3(x), \mu_4(x)) \), then our conclusion will be \( A \cap B = \phi \)

4. \( A^c = \{ x, \mu_1(x), \mu_2(x); x \in U \}^c \)
\[
= \{ x, \mu_2(x), 0; x \in U \} \cup \{ x, 1, \mu_1(x); x \in U \}
\]
5. If \( D = \{ x, \mu(x), 0; x \in U \} \)

then \( D^c = \{ x, 1, \mu(x); x \in U \} \) for all \( x \in U \).
II. Proposition: For fuzzy sets $A, B, C$ over the same universe $X$, we have the following proposition
1. $A \subseteq B, B \subseteq C \implies A \subseteq C$
2. $A \cap B \subseteq A, A \cap B \subseteq B$
3. $A \subseteq A \cup B, B \subseteq A \cup B$
4. $A \subseteq B \implies A \cap B = A$
5. $A \subseteq B \implies A \cup B = B$

III. Proposition:
Let $\tau = \{A_i : i \in I\}$ be a collection of fuzzy sets over the same universe $U$. Then
1. $\bigcup_i A_i = \{x, \max(\mu_{i_1}), \min(\mu_{i_2}); x \in U\}$
2. $\bigcap_i A_i = \{x, \min(\mu_{i_1}), \max(\mu_{i_2}); x \in U\}$

IV. Proposition:
Let $\tau = \{A_i : i \in I\}$ be a collection of fuzzy sets over the same universe $X$. Then
1. $\{\bigcup_i A_i\}^C = \bigcap_i A_i^C$
2. $\{\bigcap_i A_i\}^C = \bigcup_i A_i^C$

V. Proposition:
For a fuzzy set $A = \{x, \mu(x), \gamma(x); x \in U\}$.
$(A^C)^C = A$.

VI. Propositions:
For a fuzzy set $A$
1. $A \cap A^C = \emptyset$
2. $A \cup A^C = U$

VII. Fuzzy Topology
A fuzzy topology on a nonempty set $X$ is a family $\tau$ of fuzzy set in $X$ satisfying the following axioms

(T1) $0_X, 1_X \in \tau$

(T2) $G_i \cap G_2 \in \tau$, for any $G_1, G_2 \in \tau$

(T3) $\bigcup G_i \in \tau$, for any arbitrary family $\{G_i : G_i \in \tau, i \in I\}$.

In this case the pair $(X, \tau)$ is called a fuzzy topological space and any fuzzy set in $\tau$ is known as fuzzy open set in $X$ and clearly every element of $\tau^C$ is said to be fuzzy closed set.

Here $1_X = \{x, 1, 0; x \in X\}$ and $0_X = \{x, \mu(x), \mu(x); x \in X\}$.

VIII. Open cover
Let $(X, \tau)$ be fuzzy topology. Consider a family of fuzzy open sets $\{\{x, \mu(x), \lambda(x); x \in X, i \in I\}\}$ in $X$ satisfying the condition $\bigcup\{x, \mu(x), \lambda(x); x \in X, i \in I\} = X$, then it is said to be open cover of $X$. A finite subfamily of fuzzy open cover $\{\{x, \mu(x), \lambda(x); x \in X, i \in I\}\}$ of $X$, which is also a fuzzy open cover of $X$ is called a finite sub cover of $\{\{x, \mu(x), \lambda(x); x \in X, i \in I\}\}$. 
IX. Fuzzy compact space

A fuzzy topological space \((X, \tau)\) is called fuzzy compact if and only if every fuzzy open cover of \(X\) has a finite sub cover.

Corollary: From our definition of compact space we can give following corollary

A fuzzy set \(A=\{x, \mu_A(x), \lambda_A(x); x \in X, i \in J\}\) in fuzzy topological space \((X,\tau)\) is fuzzy compact if and only if for each family \(C=\{c_i, i \in J\}\) of fuzzy open sets in \(X\) with properties \(\mu_A(x)\leq \max_{i \in J} \mu_i\) and \(\lambda_A\geq \min_{i \in J} \lambda_i\) there exists a finite subfamily \(\{c_i, i=1, 2, 3, ..., n\}\) of \(C\) such that \(\mu_A(x)\leq \max_{i=1}^n (\mu_i)\) and \(\lambda_A\geq \min_{i=1}^n (\lambda_i)\).

Conclusion: In this paper new definition of fuzzy compact space have been discussed. Actually it is believed that the union of a fuzzy set and its complement may not be equal to the whole universal set but it is not exactly true. When we expressed fuzzy set on the basis of reference function it is seen that the union of a fuzzy set and its complement is equal to the whole universal set. By taking this idea we try to give new definition of fuzzy compact space on the basis of reference function so that our idea can help farther work of fuzzy topology.

REFERENCES