AVERAGE NUMBER OF REAL ZEROS OF RANDOM FRACTIONAL POLYNOMIAL

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Abstract: Let $a_0, a_1, \ldots$ be a sequence of mutually independent, identical standard normally distributed random variables. In this paper, the average number of real zeros of the random fractional polynomial $\sum_{k=0}^{n} a_k x^{\alpha k} (\alpha = 1/2)$, for large $n$ is obtained. Further it is proved that this average $EN(0,\infty)$ is asymptotic to $\frac{1}{\pi} \log n$.

Key words: Random variables, Random fractional polynomial, Normal distribution.

INTRODUCTION

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of non-integer orders called fractional derivatives and integrals. Recently, it is noticed that more researchers are concentrating on the fractional differential equations and its applications. The references [8, 10, 15, 16] have analytical derivations and discussions on its applications. Though the applications of fractional differential equations are vast, one of the basic examples are described below.

Random algebraic polynomials arise in the study of difference and differential equations with random coefficients. Consider the random ordinary differential operator of the form

$$T(w)[x(t)] = \sum_{k=0}^{n} a_n(w) \frac{d^{k\alpha}(x)}{dt^{k\alpha}}, \alpha \in (0,1)$$

(1.1)

In this case, the associated characteristic polynomial is a random algebraic polynomial of the form

$$F_n(\lambda, \omega) = \sum_{k=0}^{n} a_n(\omega) \lambda^{k\alpha}$$

(1.2)

To study the solution of $T(\omega)[x(t)]=0$, one has to study the solution of $F(\lambda,x)=0$. 

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In particular for $k=2$ and $\alpha=1/2$, $a_0(\omega)=1$ almost surely, then random fractional differential equation is

$$\frac{dx}{dt} + a_1(\omega)\frac{d^2}{dt^2} + a_2(\omega) = 0$$

(1.3)

To study (1.3) one have to know the solution properties of fractional random algebraic polynomials

$$\lambda + a_1(\omega)\lambda^{1/2} + a_2(\omega) = 0$$

Therefore the presented result in this article is to initiate the study of fractional random algebraic polynomial.

Consider the algebraic equation,

$$a_0 + a_1x^\alpha + a_2x^{2\alpha} + \ldots + a_{n-1}x^{(n-1)\alpha} = 0 \quad (0 \leq x < \infty)$$

(1.4)

Where the $a_i$'s ($i=0,1,\ldots,n-1$) are independent random variables, assuming real values only and $\alpha \in (0,1)$. In particular if $\alpha=1/2$, equation (1.4) yields,

$$f(x) = \sum_{k=0}^{n-1} a_k x^k = a_0 + a_1x^{1/2} + a_2x + \ldots + a_{n-1}x^{(n-1)/2} \quad (0 \leq x \leq \infty)$$

(1.5)

When $\alpha=1$, that is for the polynomial $\sum a_k x^k$, where $a_k$'s ($k=0,1,\ldots,n-1$) are independent and normally distributed random variables with mean 0 and variance 1 is estimated in Kac[5] and $\text{EN}_n(-\infty, \infty)$ is $\frac{2}{\pi} \log n$. This relation is known as Kac's result.

Further in [6], it is shown that the result is unaltered if the normally distributed random variables are replaced by an uniformly distributed random variables.

Stevens [12] treated the case when the $a_i$'s ($i=1,2,\ldots,n$) are independent random variables with $\alpha=1$ and showed that the average number of zeros is asymptotic to $\frac{2}{\pi} \log n$.

When $E(a_i) = 0, E(a_i^2) = 1, E(a_i^4) < B$

$$\frac{d}{dy} p(a_i < y) < \frac{B}{1 + y^{1/2}}$$

and
for some finite B and large ‘n’.

When the random variables are dependent and normally distributed with mean 0 and variance 1, Sambandham[10] estimated the average no of real zeros of $\sum a_k x^k$ and this average is asymptotic to $\frac{1}{\pi} \log n$.

Further Sambandham[11] estimated the average number of real zeros of the polynomial $\sum a_k x^k$ for large ‘n’ and this $EN_n(-\infty, \infty)$ asymptotic to $\frac{1}{2\pi} [1 + (2p + 1)^{1/2}] \log n$.

Das [2] has studied the case, when the coefficients of the random variables are independent and standard normally distributed, then $EN_n(-\infty, \infty)$ of real zeros of

$$\sum a_k x^k$$ is $\frac{1}{\pi} [1 + (2p + 1)^{1/2}] \log n$. Under the same condition Das[3] estimated the $EN_n(-\infty, \infty)$ of maxima of $\sum a_k x^k$ is $\frac{1}{2\pi} [1 + (3)^{1/2}] \log n$.

B.F. Logan [8] and Shepp[8] estimated the $EN_n(-\infty, \infty)$ of real zeros of $\sum \xi_j t^j$ as c logn,where the coefficients are independent random variables with common characteristic function $\exp(-|z|^\alpha), 0 < \alpha \leq 2$.The constant c=c(\alpha).For $\alpha=2$,it reduces to Kac’s result.

In this article, the real polynomials are only considered. When $0 < \alpha < 1$, $(-x^n)$ will not be real. To consider real polynomial, it is assumed that $0 < x < \infty$.

**Theorem (1.1)**

If the random variables $a_i$’s (i=0,1,2…n-1) are independent and standard normally distributed, the average number of real zeros, $E(N_n)$ of the equation (1.5) is given by the following formula,

$$EN_n(0, \infty) = \frac{1}{\pi} \int_0^{1/2} \frac{(1 + x^{2n} + 2x^n(n^2 - 1) - n^2 x^{n+1} - n^2 x^{n+1})^{1/2}}{x^2(1-x)(1-x^n)} dx \quad (1.6)$$

and the asymptotic relation is
and the estimate,
\[ \frac{1}{\pi} \log n \leq EN_n(0, \infty) \leq \frac{1}{\pi} \log n + 0.2 \]  
(1.8)

Where \( E(N_n) \) denotes the average number of real zeros of the equation (1.5).

### 3 Proof of the Main Theorem (1.1)

By Kac's formula [5],
\[ EN_n(a, b) = \frac{1}{\pi} \int_a^b \frac{(AC - B^2)^{1/2}}{A} \, dx \]  
(2.1)

Let \( \Delta = AC - B^2 \)  
(2.2)

Where
\[ A = \sum_{k=0}^{n-1} x^k \]
\[ B = \sum_{k=0}^{n-1} \frac{k}{2} x^{k-1}, \text{ and} \]
\[ C = \sum_{k=0}^{n-1} kx^{k-1} \]
for \( 0 < x < 1 \).

Equivalently
\[ A = \frac{1 - x^n}{1 - x} \]
\[ B = \frac{1}{2} \left[ \frac{1 - x^n - nx^{n-1}(1-x)}{(1-x)^2} \right], \text{ and} \]
\[ C = \frac{1}{4x(1-x)^3} \left[ (1-x^n)(1+x) - n^2 x^{n-1}(1-x)^2 - 2nx^2(1-x) \right] \]

By Cauchy's inequality in Sambandham[10], in (a,b), \( \Delta > 0 \).

Substituting the value of A, B and C given in the system of equations (2.4) in equation (2.2) yields
\[ \Delta = \frac{1 + 2x^n(n^2 - 1) + x^{2n} - n^2x^{n-1} - n^2x^{n+1}}{4x(1-x)^3} \]  

(2.5)

and by an analytical computation

\[ EN_n(0,1) = \frac{1}{2\pi} \int_0^{\infty} \frac{(1 + x^{2n} + 2x^n(n^2 - 1) - n^2x^{n-1} - n^2x^{n+1})^{\frac{1}{2}}}{x^2(1-x)^2(1-x^n)} dx \]

(2.6)

The transformation \( \frac{1}{x^2} \rightarrow x \) implies that the number of real zeros in the interval \((0, 1)\) is same as in the interval \((1, \infty)\).

Therefore,

\[ EN_n(0,\infty) = \frac{1}{\pi} \int_0^{\frac{1}{1-x^n}} \frac{[1 - G_n^2(x)]^{\frac{1}{2}}}{x^2(1-x)} dx \]

(2.7)

3 ASYMPTOTIC VALUE OF E(N_n)

Consider

\[ EN_n(0,\infty) = \frac{1}{\pi} \int_0^{\frac{1}{1-x^n}} \frac{[1 - G_n^2(x)]^{\frac{1}{2}}}{x^2(1-x)} dx \]  

(3.1)

Where

\[ G_n(x) = \frac{nx^{r_0} - nx^{r_1}}{1 - x^n} \]  

(3.2)

Then

\[ G_n(x) > x^{r_2} \], \quad 1 - G_n^2(x) < (1 + x^{r_2})(1 - x^{r_2}) \], and
1 - G^2_n(x) \leq (1 - x^{\frac{n+1}{2}}) \quad (3.3)

Suppose f(x) is differentiable on [0,1] applying mean value theorem for f(x) on [x,1] gives,

\[ f(1) - f(x) = f'(\theta)(1-x) \quad \text{for } x, \theta \in [0,1] \quad (3.4) \]

Set \( f(x) = 1 - x^{\frac{n+1}{2}} \),

\[ \Rightarrow 1 - G^2_n(x) = (1-x) \left[ \frac{n-1}{2} - \frac{\theta^{n-3}}{2} \right] \]

\[ \Rightarrow 1 - G^2_n(x) = (1-x)\left(\frac{n-1}{2}\right) \quad \text{for } x < \theta < 1 \]

\[ \Rightarrow \frac{[1 - G^2_n(x)]^{\frac{1}{2}}}{1-x} < \frac{(\frac{n-1}{2})^{\frac{1}{2}}}{1} \quad (0 \leq x \leq 1) \quad (3.5) \]

On the other hand, \( G_n(x) \to 0 \) as \( n \to \infty \), so \( G^2_n(x) \to 0 \) as \( n \to \infty \).

Then \( 1 - G^2_n(x) \leq 1 \),

\[ \Rightarrow [1 - G^2_n(x)]^{\frac{1}{2}} \leq 1 \]

\[ \Rightarrow \frac{[1 - G^2_n(x)]^{\frac{1}{2}}}{1-x} \leq \frac{1}{1-x} \quad (0 \leq x \leq 1) \quad (3.6) \]

and

\[ \int_0^{\frac{1}{2}} \frac{[1 - G^2_n(x)]^{\frac{1}{2}}}{x^2(1-x)} \, dx < \int_0^{\frac{1}{2}} \frac{1}{x^2(1-x)} \, dx + \int_0^{\frac{1}{2}} \frac{(\frac{n-1}{2})^{\frac{1}{2}}}{x^2(1-x)} \, dx \quad (3.7) \]

After integration,

\[ \int_0^{\frac{1}{2}} \frac{dx}{x^2(1-x)} = \log(2 - \frac{1}{n}) + \log n \quad (3.8) \]
As \( n \to \infty \), the second integral tends to zero,

\[
\int_{(1/n)^2}^{1} \frac{1}{2} \left( \frac{n-1}{2} \right)^{1/2} dx \to 0
\]

that is

\[
\int_{(1/n)^2}^{1} \frac{1}{2} \frac{n-1}{2} x^2 (1-x) dx \to 0
\]

Substituting the results (3.8) and (3.9) in the equation (3.7) yields

\[
\int_{0}^{1} \frac{1}{x^2 (1-x)} \left[ 1 - G_n^2(x) \right]^{1/2} dx < \log(2 - \frac{1}{n}) + \log n
\]

(3.10)

From the equation (3.1),

\[
EN_n(0, \infty) < \frac{1}{\pi} \log n + \frac{1}{\pi} \log(2 - \frac{1}{n})
\]

(3.11)

For large values of \( n \),

\[
EN_n(0, \infty) < \frac{1}{\pi} \log n + 0.2
\]

(3.12)

To obtain the lower estimate, let \( \epsilon \) and \( \delta \) be real quantities with \( 0 < \epsilon < 1 \) and \( 0 < \delta < 1 \)

\[
G_n^2(x) \to 0 \quad \text{when} \quad n \to \infty.
\]

So for sufficiently large \( n \),

\[
G_n^2(x) < \epsilon
\]

(3.13)

\[
\int_{0}^{1} \frac{1}{x^2 (1-x)} \left[ 1 - G_n^2(x) \right]^{1/2} dx > \int_{0}^{(1-n^{\epsilon-1})^2} \frac{1}{x^2 (1-x)} dx
\]

(3.14)

But in

\[
[0, (1-n^{\delta-1})^2]
\]

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Using equation (3.13),

\[
\int_0^1 \frac{(1 - G_n^2(x))^\frac{1}{2}}{x^\frac{3}{2}(1-x)} \, dx > \int_0^1 \frac{(1 - G_n^2(x))^\frac{1}{2}}{1 - x^\frac{3}{2}} \, dx > \int_0^1 \frac{(1 - \varepsilon)^\frac{1}{2}}{1 - x^\frac{3}{2}} \, dx
\]

(3.15)

After suitable transformation,

\[
\int_0^1 \frac{(1 - G_n^2(x))^\frac{1}{2}}{x^\frac{3}{2}(1-x)} \, dx > (1 - \varepsilon)^\frac{1}{2} (1 - \delta) \log n
\]

(3.16)

That is

\[
EN_n(0, \infty) > \frac{1}{\pi} (1 - \varepsilon)^\frac{1}{2} (1 - \delta) \log n
\]

(3.17)

From the equations (3.12) and (3.17) the asymptotic formula

\[
EN_n(0, \infty) \asymp \frac{1}{\pi} \log n
\]

(3.18)

is obtained.

Equation (3.18) represents the asymptotic value of the real zeros of the random fractional polynomial for 0<x<\infty. This proves Theorem 1.1.

REFERENCES


