FIXED POINT AND COMMON FIXED POINT RESULTS FOR EXPANSIVE MAPPINGS IN $G_b$-CONE METRIC SPACES

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Abstract: Very recently, Manoj Ughade and R. D. Daheriya [18] introduced the concept of $G_b$-cone metric and established some fixed point and common fixed point theorems for self-mappings satisfying various contractive conditions in $G_b$-cone metric spaces without the assumption of normality. In this paper, we present some fixed point theorem for self-mappings satisfying expansive type condition $G_b$-cone metric spaces without the assumption of normality. Moreover, some examples are provided to illustrate the usability of the obtained results.

Keywords and Phrases: $G_b$-cone metric spaces; normal cones; non-normal cones; expansive mappings; fixed points.

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1. INTRODUCTION

Metric spaces play significant in mathematics and applied sciences. So, some authors have tried to give generalizations of metric spaces in several ways. The notion of D-metric space is a generalization of usual metric spaces and it is introduced by Dhage [1-4]. Mustafa and Sims [5-6] have shown that most of the results concerning Dhage’s D-metric spaces are invalid. In [5-6], they introduced an improved version of the generalized metric space structure which they called G-metric spaces. Beg, I, Abbas, M, Nazir [13] introduced G-cone metric space and established some new fixed point theorems. Recently, Asadollah Aghajani, Mujahid Abbas and Jamal Rezaei Roshan [12] introduced $G_b$-metric space and established common fixed point of generalized weak contractive mappings in partially ordered $G_b$-metric spaces. Very recently, Manoj Ughade and R. D. Daheriya [18] introduced the concept of $G_b$-cone metric and some established some fixed point and common fixed point theorems for self-mappings satisfying various contractive conditions in $G_b$-cone metric spaces without the assumption of normality. The study of expansive mappings is a very interesting research area in fixed point theory. In 1984, Wang et.al [27] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko [26] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. Chintaman and Jagannath [28] introduced several meaningful fixed point theorems for one expanding mapping.
In this paper, we present some fixed point theorem for self-mappings satisfying expansive type condition $G_b$-cone metric spaces without the assumption of normality. These results improve and generalize some important known results. Illustrative some examples to highlight the realized improvements is also furnished.

2. DEFINITION AND PRELIMINARIES

We collect some relevant definitions and fundamental results for our further use.

Definition 2.1 (see [12]) Let $X$ be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \rightarrow \mathbb{R}^+$ satisfies:

(GB1). $G(x,y,z) = 0$ if $x = y = z$,

(GB2). $0 < G(x,x,y)$, $\forall x,y \in X$, with $x \neq y$,

(GB3). $G(x,y) \leq G(x,y,z)$, $\forall x,y,z \in X$ with $y \neq z$.

(GB4). $G(x,y,z) = G(p(x,y))$ (Symmetry).

(GB5). $G(x,y,z) \leq s\left(G(x,a,a) + G(a,y,z)\right)$, $\forall x,y,z,a \in X$ (Rectangle inequality).

Then $G$ is called a generalized b-metric, or, more specially, $G_b$-metric on $X$, and the pair $(X,G)$ is called a $G_b$-metric space. If $s = 1$, then $G$ is called a generalized metric on $X$, and the pair $(X,G)$ is called a $G$-metric space.

Definition 2.2 Let $E$ be a real Banach space, a subset of $P$ of $E$ is called a cone if and only if:

a) $P$ is closed, non empty and $P \neq \{0\}$.

b) $a,b \in \mathbb{R}$, $a,b \geq 0$, $x,y \in P \Rightarrow ax + by \in P$, more generally, if $a,b,c \in \mathbb{R}$, $a,b,c \geq 0,x,$

$y,z \in P \Rightarrow ax + by + cz \in P$

c) $x \in P$ and $-x \in P \Rightarrow x = 0, e$, i.e., $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. A cone $P \subset E$ is called normal if there is a number for all $K > 0$ such that for all $x,y \in E, 0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying the above inequality is called the normal constant of $P$, while $x \ll y$ stands for $y - x \in \text{int}P$. (int$P$ denotes the interior of $P$), while $x \prec y$ means that $x \leq y$ but $x \neq y$.

Very recently, Ughade and Daheriya [18] introduced the concept of $G_b$-metric space as follows.

Definition 2.3 Let $X$ be a nonempty set and $E$ be a real Banach space equipped with the partial ordering $\leq$ with respect to the cone $P$. A vector-valued function $G : X \times X \times X \rightarrow E$ is said to be a generalized cone b-metric function on $X$ with the constant $s \geq 1$ if the following conditions are satisfied:

(GBC1). $G(x,y,z) = 0$ if $x = y = z$,

(GBC2). $0 < G(x,x,y)$, whenever $x \neq y$, $\forall x,y \in X$,

(GBC3). $G(x,y) \leq s\left(G(x,y,z)\right)$, whenever $y \neq z$, $\forall x,y,z \in X$, 

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Then the pair \((X, G)\) is called a generalized \(G_b\)-cone metric space or, more specifically, a \(G_b\)-cone metric space. Obverse that if \(s = 1\) the ordinary rectangle inequality in a generalized cone metric space is satisfied; however, it does not hold true when \(s > 1\). Thus the class of \(G_b\)-cone metric spaces are effectively larger than that of ordinary \(G\)-cone metric spaces. That is, every \(G\)-cone metric space is a \(G_b\)-cone metric space, but the converse need not be true. Therefore, it is obvious that \(G_b\)-cone metric spaces generalize \(G\)-metric spaces and \(G\)-cone metric spaces.

The following examples, definitions and results are required in the sequel which can be found in [18].

**Example 2.4** Let \(X = \mathbb{R}\) and \(E = \mathbb{R}^2\). Define \(G: X \times X \times X \to E\) by

\[
G(x, y, z) = \max\{(|x - y|^p, \alpha|x - y|^p), (|y - z|^p, \alpha|y - z|^p), (|z - x|^p, \alpha|z - x|^p)\}
\]

\(\forall \ x, y, z \in X\), where \(\alpha \geq 0\) and \(p > 1\) are two constants. Then \((X, G)\) is a \(G_b\)-cone metric space on \(X\), but not a \(G\)-cone metric.

**Example 2.5** Let \(X = [0, +\infty)\) and \(E = C^1_{[0,1]}\) with the norm \(\|x\| = \|x\|_{\infty} + \|x'\|_{\infty}\) and consider \(P = \{x \in E: x(t) \geq 0\} on 0,1\). Define \(G:X \times X \times X \to E\) by

\[
G(x, y, z) = \max\{|x - y|^p, |y - z|^p, |z - x|^p\} \varphi,
\]

\(\forall \ x, y, z \in X\), with \(p \geq 1\), where \(\varphi: [0,1] \to \mathbb{R}\) such that \(\varphi(t) = e^t\). Then \((X, G)\) is a complete \(G_b\)-cone metric space with the coefficient \(s = 2p^{-1}\). If we define \(G: X \times X \times X \to E\) by

\[
G(x, y, z) = (|x - y|^2 + |y - z|^2 + |z - x|^2)\varphi,
\]

\(\forall \ x, y, z \in X\). Then \((X, G)\) is not a \(G_b\)-cone metric space. However,

\[
G(x, y, z) = \max\{|x - y|^2, |y - z|^2, |z - x|^2\}\varphi
\]

is a \(G_b\)-cone metric space on \(X\) with \(s = 2\).

**Example 2.6** Let \(X = [0,1]\) and \(E = C^1_{[0,1]}\) with the norm \(\|x\| = \|x\|_{\infty} + \|x'\|_{\infty}\) and consider \(P = \{x \in E: x(t) \geq 0\) on \([0,1]\). Define \(G: X \times X \times X \to E\) by

\[
G(x, y, z) = (|x - y| + |y - z| + |z - x|)^2\varphi,
\]

\(\forall \ x, y, z \in X\), where \(\varphi: [0,1] \to \mathbb{R}\) such that \(\varphi(t) = e^t\). Then \((X, G)\) is a complete \(G_b\)-cone metric space with the coefficient \(s = 2\).

**Definition 2.7** A \(G_b\)-cone metric space \((X, G)\) is said to symmetric if \(\forall \ x, y \in X, G(x, y, y) = G(y, x, x)\).

Let \((X, G)\) be a \(G_b\)-cone metric space, define \(d_{G_b}: X \times X \to E\) by

\[
d_{G_b}(x, y) = G(x, y, y) + G(y, x, x).
\]
Then $(X, d_{G_b})$ is a cone $b$-metric space. It can be noted that

$$G(x, y, y) \leq \frac{2s}{2s+1} d_{G_b}(x, y).$$

Obverse that if $s = 1$, that is, $G$ be a $G$-cone metric on $X$, then

$$G(x, y, y) \leq \frac{2}{3} d_G(x, y)$$

If $X$ is a symmetric $G_b$-cone metric space, then

$$d_{G_b}(x, y) = 2G(x, y, y).$$

**Definition 2.8** Let $(X, G)$ be a $G_b$-cone metric space. A sequence $(x_n)$ in $X$ is said to be:

1) a $G_b$-cone Cauchy sequence if, for every $c \in E$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that for all $n, m, l > N, G(x_n, x_m, x_l) \ll c$.

2) a $G_b$-cone convergent sequence if, for every $c \in E$ with $\theta \ll c$, there is $N \in \mathbb{N}$ such that for all $m, n > N$, $G(x_n, x_m, x) \ll c$ for some fixed $x$. Here $x$ is called the $G_b$-limit of $(x_n)$ and is denoted by $G_b \lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

**Definition 2.9** A $G_b$-cone metric space $X$ is said to be a $G_b$-complete cone metric space, if every $G_b$-cone Cauchy sequence in $X$ is $G_b$-cone convergent in $X$.

**Lemma 2.10** Let $(X, G)$ be a $G_b$-cone metric space. Then for $c \in E$ with $c \gg \theta$, there is $\delta > 0$ such that $\|x\| < \delta$ implies $c - x \in \text{int} P$.

**Lemma 2.11** Let $(X, G)$ be a $G_b$-cone metric space, $P$ be a normal cone with normal constant $K$. Let $(x_n)$ be a sequence in $X$. Then $(x_n)$ is $G_b$-cone convergent to $x$ if and only if $G(x_m, x_n, x) \to \theta$ as $m, n \to +\infty$.

**Proposition 2.12** Let $(X, G)$ be a $G_b$-cone metric space, $P$ be a normal cone with normal constant $K$, then the following are equivalent:

1. $(x_n)$ is $G_b$-cone convergent to $x$.
2. $G(x_m, x_n, x) \to \theta$, as $n \to +\infty$.
3. $G(x_m, x_n, x) \to \theta$, as $m \to +\infty$.
4. $G(x_m, x_n, x) \to \theta$, as $m, n \to +\infty$.

**Lemma 2.13** Let $(X, G)$ be a complete $G_b$-cone metric space with the coefficient $s \geq 1$, $P$ be a normal cone with normal constant $K$. Let $(x_n)$ be a sequence in $X$. If $(x_n)$ $G_b$-cone converges to $x$ and also $(x_n)$ $G_b$-cone converges to $y$, then $x = y$. That is the limit of $(x_n)$ is unique.
Proposition 2.14 Let $(X, G)$ be a $G_n$-cone metric space, $P$ be a normal cone with normal constant $K$. Then sequence $(x_n)$ is $G_n$-cone Cauchy if and only if $G(x_n, x_m, x_l) 	o \theta$, as $n, m, l \to +\infty$.

Lemma 2.15 Let $(X, G)$ be a $G_n$-cone metric space, $(x_n)$ be a sequence in $X$. If $(x_n)$ is a $G_n$-cone convergent to $x$ in $X$, then $(x_n)$ is a $G_n$-cone Cauchy sequence in $X$.

Proposition 2.16 Let $(X, G)$ be a $G_n$-cone metric space, then the following are equivalent:

1. $(x_n)$ is $G_n$-cone Cauchy in $X$.
2. For every $c \in E$ with $c \gg \theta$, there is $N \in \mathbb{N}$ such that for all $n, m > N$, $G(x_n, x_m, x_m) \ll c$.

Proposition 2.17 Let $(X, G)$ be a $G_n$-cone metric space, $P$ be a normal cone with normal constant $K$. Let $(x_n)$ and $(y_n)$ be two sequences in $X$ and suppose that $x_n \to x, y_n \to y$ as $n \to +\infty$. Then $G(x_n, x_n, y_n) \to s^2G(x, x, y)$ as $n \to +\infty$.

Proposition 2.18 Let $(X, G)$ be a $G_n$-cone metric space, $P$ be a normal cone with normal constant $K$. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Lemma 2.19 Let $(x_n)$ be a sequence in a $G_n$-cone metric space $(X, G)$ with the coefficient $s \geq 1$ relative to a solid cone $P$ such that

$$G(x_n, x_{n+1}, x_{n+1}) \leq sG(x_{n-1}, x_n, x_n)$$

where $\lambda \in \left[0, \frac{1}{s}\right]$ and $n = 1, 2, \ldots$. Then $(x_n)$ is a Cauchy sequence in $(X, G)$.

Lemma 2.20 [26] For the case of non normal cones, we have the following properties.

(PT1). If $u \ll v$ and $v \ll w$, then $u \ll w$.
(PT2). If $u \ll v$ and $v \ll w$, then $u \ll w$.
(PT3). If $u \ll v$ and $v \ll w$, then $u \ll w$.
(PT4). If $\theta \ll u \ll c$ for each $c \in \text{int}P$, then $u = \theta$.
(PT5). If $a \ll b + c$ for each $c \in \text{int}P$, then $a \ll b$.
(PT6). If $E$ be a real Banach space with a cone $P$, and if $a \ll \lambda a$, where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.
(PT7). If $c \in \text{int}P$, $a_n \in E$ and $a_n \to \theta$, then there exists an $n_0$ such that, for all $n > n_0$, we have $a_n \ll c$.

3. MAIN RESULTS

In this section, we will present some fixed point and common fixed point theorems for expansive mappings in the setting of $G_n$-cone metric spaces. Furthermore, we will give examples to support our main results. Throughout this section, we not impose the normality condition for the cones, but the only assumption is that the cone $P$ is solid, that is, $\text{int}P \neq \emptyset$.

Now, our first main results as follows.
Theorem 3.1 Let \((X, G)\) be a complete \(G_{\alpha}\)-cone metric space with the coefficient \(s \geq 1\) relative to a solid cone \(P\). Assume that the mapping \(T: X \to X\) is a surjection and satisfies

\[
G(Tx, Ty, Tz) \leq \lambda G(x, y, z), \quad \forall x, y, z \in X,
\]

where \(\lambda > s\). Then \(T\) has a fixed point.

Proof Let \(x_0 \in X\), since \(T\) is surjection on \(X\), then there exists \(x_1 \in X\) such that \(Tx_0 = Tx_1\). By continuing this process, we get

\[
x_n = Tx_{n+1}, \quad \forall n \in \mathbb{N} \cup \{0\}.
\]

In case \(x_{n_0} = x_{n_0+1}\) for some \(n_0 \in \mathbb{N} \cup \{0\}\), then it is clear that \(x_{n_0}\) is a fixed point of \(T\). Now assume that \(x_n \neq x_{n-1}\) for all \(n\). Consider,

\[
G(x_{n-1}, x_n, x_n) = G(Tx_n, Tx_{n+1}, Tx_{n+1})
\]

Now by (1) and definition of the sequence

\[
G(x_{n-1}, x_n, x_n) = G(Tx_n, Tx_{n+1}, Tx_{n+1})
\]

and so

\[
G(x_n, x_{n+1}, x_{n+1}) \leq \frac{1}{\lambda} G(x_{n-1}, x_n, x_n)
\]

for all \(n \in \mathbb{N} \cup \{0\}\), where \(h = \frac{1}{\lambda} < \frac{1}{s}\). So, by Lemma 2.19, \((x_n)\) is Cauchy sequences in \((G, X)\). Since \((X, G)\) is a complete \(G_{\alpha}\)-cone metric space, there exists \(x^* \in X\) such that \(x_n \to x^*\) and then for given \(c > 0\), we have \(G(x_n, x^*, x^*) < \lambda c\) for all \(n > n_0\).

Now since \(T\) is surjective map. So there exists a point \(p\) in \(X\) such that \(x^* = Tp\). Consider from (1), we have

\[
G(x_n, x^*, x^*) = G(Tx_n, Tp, Tp)
\]

That is,

\[
G(x_{n+1}, p, p) \leq \frac{1}{\lambda} G(x_n, x^*, x^*) < c
\]

for each \(n > n_0\). Therefore, \(x_{n+1} \to p\). From Lemma 2.13, we deduce that \(p = x^*\) and so \(x^*\) is fixed point of \(T\). Now we show that the fixed point is unique. If there is another fixed point \(y^*\), by the given condition (1),

\[
G(x^*, y^*, y^*) = G(Tx^*, Ty^*, Ty^*)
\]

That is,

\[
G(x^*, y^*, y^*) < k G(x^*, y^*, y^*)
\]
where $k = \frac{1}{\lambda} < \frac{1}{\lambda} < 1$. By Lemma 2.20 (PT6), we have $x^* = y^*$. The proof is completed. ■

Next example illustrates Theorem 3.1.

Example 3.2 Let $X = [0, +\infty)$ and $E = C_0^2[0,1]$ with the norm $\|x\| = \|x\|_{\infty} + \|x'\|_{\infty}$ and consider $P = \{x \in E : x(t) \geq 0 \text{ on } 0,1\}. Define G:X \times X \rightarrow E$ by

$$G(x, y, z) = \max\{|x - y|^p, |y - z|^p, |z - x|^p\} \varphi,$$

for all $x, y, z \in X$, with $p \geq 1$, where $\varphi: [0,1] \rightarrow \mathbb{R}$ such that $\varphi(t) = e^t$. Then $(X, G)$ is a complete $G_b$-cone metric space with the coefficient $s = 2^{p-1}$. Let us define $T : X \rightarrow X$ as $Tx = \frac{x}{2}$ for all $x \in X$. Therefore

$$G(Tx, Ty, Tz) = \max\{|Tx - Ty|^p, |Ty - Tz|^p, |Tz - Tx|^p\} e^t,$$

$$= \max\{|\frac{x}{2} - \frac{y}{2}|^p, |\frac{y}{2} - \frac{z}{2}|^p, |\frac{z}{2} - \frac{x}{2}|^p\} e^t,$$

$$= \left(\frac{2}{3}\right)^p \max\{|x - y|^p, |y - z|^p, |z - x|^p\} e^t,$$

$$= 2^p \max\{|x - y|^p, |y - z|^p, |z - x|^p\} e^t,$$

$$= \lambda G(x, y, z).$$

where $\lambda = 2^p > s = 2^{p-1}$. Here $0 \in X$ is the unique fixed point of $T$.


Definition 3.3 Let $S$ and $T$ be two self-mappings on a nonempty set $X$. Then $S$ and $T$ are said to be weakly compatible if they commute at all of their coincidence points; that is, $Sx = Tx$ for some $x \in X$ and then $STx = TSx$.

We need the following definition:

Lemma 3.4 (see [32]) Let $S$ and $T$ be weakly compatible self-mappings of a nonempty set $X$. If $S$ and $T$ have a unique point of coincidence $w = Sx = Tx$, then $w$ is the unique common fixed point of $S$ and $T$. ■

Now we establish that common fixed points for a pair of two weakly compatible self-mappings satisfying expansive condition are proved in the frame of $G_b$-cone metric spaces.

Theorem 3.5 Let $(X, G, s)$ be a complete $G_b$-cone metric space. Let $S$ and $T$ be a weakly compatible self-mappings of $X$ and $T(X) \subseteq S(X)$. Suppose that there exist $k > s$ such that

$$G(Sx, Sy, Sz) \geq k G(Tx, Ty, Tz), \forall x, y, z \in X.$$

(7)
If one of the subspaces $T(X)$ or $S(X)$ is complete, then $S$ and $T$ have a unique common fixed point in $X$.

**Proof** Let $x_0 \in X$. Since $T(X) \subseteq S(X)$, choose $x_1$ such that $y_1 = Sx_1 = Tx_0$. In general choose $x_{n+1}$ such that $y_{n+1} = Sx_{n+1} = Tx_n$, then from condition (7),

$$G(y_{n+2}, y_{n+3}, y_{n+1}) = G(Tx_n, Tx_{n+1}, Tx_{n+2})$$

$$\leq \frac{1}{k} G(Sx_n, Sx_{n+1}, Sx_{n+2})$$

$$= \frac{1}{k} G(Tx_{n+1}, Tx_n, Tx_{n-1})$$

$$= \frac{1}{k} G(y_n, y_{n+1}, y_{n+2})$$

$$= \lambda G(y_n, y_{n+1}, y_{n+2})$$

where $\lambda = \frac{1}{k} < \frac{1}{2}$. So, by Lemma 2.19, $(y_n)$ is Cauchy sequences in $(G, X)$. Since $(X, G)$ is a complete $G_0$-cone metric space, there exists $x^* \in X$ such that $y_n \to x^*$ that is

$$\lim_{n \to \infty} y_n = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Sx_n = x^*.$$

Then for given $c > 0$, we have $G(y_n, x^*, x^*) < kc$ for all $n > n_0$. Since $T(X)$ or $S(X)$ is complete and $T(X) \subseteq S(X)$, there exists a point $p \in X$ such that $Sp = x^*$. Now from (7),

$$G(Tp, y_n, y_n) = G(Tp, Tx_{n-1}, Tx_{n-1})$$

$$\leq \frac{1}{k} G(Sp, Sx_n, Sx_n)$$

$$= \frac{1}{k} G(x^*, y_n, y_n) < c$$

for each $n > n_0$. Therefore, $y_n \to Tp$. From Lemma 2.13, we deduce that $Tp = x^*$ and so $x^*$ is coincidence point of $T$ and $S$. Assume that there exist $u, v$ in $X$ such that $Tu = Su = v$. From (7), we have

$$G(Su, Sp, Sp) \geq k G(Tu, Tp, Tp) = k G(Su, Sp, Sp)$$

That is

$$G(Su, Sp, Sp) \leq \frac{1}{k} G(Su, Sp, Sp)$$

where $\lambda = \frac{1}{k} < \frac{1}{2} < 1$. Thus, by Lemma 2.19 (PT7), we can obtain that $G(Su, Sp, Sp) = \theta$, i.e. $v = Sp = Su = q$. Moreover, the mappings $S$ and $T$ are weakly compatible, by Lemma 3.8, we know that $q$ is the unique common fixed point of $S$ and $T$.

\[\blacksquare\]

Next example illustrates Theorem 3.5.

**Example 3.6** Let $X = [1, +\infty)$, $E = C_1[0,1]$ with the norm $\|x\| = \|x\| + \|x^\prime\|$, and consider $P = \{q \in E : q \geq 0\} \subseteq E$. Define $G: X \times X \times X \to E$ by

$$G(x, y, z) = \max\{|x - y|^2, |y - z|^2, |z - x|^2\} e^t,$$
∀ x, y, z ∈ X. Then \((X, G)\) is a complete \(G_s\)-cone metric space with the coefficient \(s = 2\), but it is not a cone metric space. We consider the functions \(S, T : X \rightarrow X\) defined by

\[
Sx = \frac{1}{2} \ln x + 1 \quad \text{and} \quad Tx = \frac{1}{6} \ln x + 1.
\]

Obviously, \(S(X) \subseteq T(X)\) is a complete subspace of \(X\). Here

\[
G(Tx, Ty, Tz) = \max\{|Tx - Ty|^2, |Ty - Tz|^2, |Tz - Tx|^2\} e^t
\]

\[
= \frac{1}{86} \max\{|\ln x - \ln y|^2, |\ln y - \ln z|^2, |\ln z - \ln x|^2\} e^t
\]

Now,

\[
G(Sx, Sy, Sz) = \max\{|Sx - Sy|^2, |Sy - Sz|^2, |Sz - Sx|^2\} e^t
\]

\[
= \max\left\{\frac{1}{12} |\ln x - \frac{1}{2} \ln y|^2, \frac{1}{12} |\ln y - \frac{1}{2} \ln z|^2, \frac{1}{12} |\ln z - \frac{1}{2} \ln x|^2\right\} e^t
\]

\[
= \frac{1}{4} \max\{|\ln x - \ln y|^2, |\ln y - \ln z|^2, |\ln z - \ln x|^2\} e^t
\]

\[
\geq k \max\left\{\frac{1}{6} |\ln x - \frac{1}{2} \ln y|^2, \frac{1}{6} |\ln y - \frac{1}{2} \ln z|^2, \frac{1}{6} |\ln z - \frac{1}{2} \ln x|^2\right\} e^t
\]

\[
= \frac{1}{6} \max\{|Tx - Ty|^2, |Ty - Tz|^2, |Tz - Tx|^2\} e^t
\]

\[
= \frac{1}{6} G(Tx, Ty, Tz)
\]

for \(2 < k \leq 9\) and (7) is satisfied. Also \(S1 = T1 \Rightarrow ST1 = TS1\), that is, the pair \((S, T)\) is weakly compatible. It is clear that the conditions of Theorem 3.5 are satisfied. Here \(x^* = 1\) is a unique common fixed point of \(S\) and \(T\).

COMPETING INTERESTS

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AUTHOR’S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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