\(\hat{g}^s\)-continuous maps in topological spaces

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Abstract: In this paper, we introduce the concepts of \(\hat{g}^s\)-continuity and \(\hat{g}^s\)-irresoluteness mappings and their characterizations.

Key words: \(\hat{g}^s\)-continuity, \(\hat{g}^s\)-irresoluteness, \(\hat{g}^s\)-open map, \(\hat{g}^s\)-closed map

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1 Introduction

The concept of generalised closed set of a topological space was introduced by N. Levine in 1970 [6]. These sets were also considered by W. Dunham and N. Levine in 1980 [3] and by W. Dunham in 1982 [4]. Since then new concepts have been introduced, studied, investigated and developed in the field of generalised closed sets by various authors. In 1991, K. Balachandran, H. Maki and P. Sundaram [1] defined a new class of mappings called generalised continuous mappings which contains the class of continuous mappings. S. Poius Missier and M. Anto studied and investigated the topological properties of \(\hat{g}^s\)-closed sets [8] by generalising the semi closed sets using \(g\)-open sets. Based on \(\hat{g}^s\)-closed sets, we continue the study of the associated functions, namely, \(\hat{g}^s\)-irresolute and \(\hat{g}^s\)-continuous functions.

2 Preliminaries

Definition 2.1 [7] A subset \(A\) of a topological space \((X, \tau)\) is called semi open if \(A \subseteq cl (int (A))\).

A subset \(A\) of a topological space \((X, \tau)\) is called semi closed if \(A^c\) is semi open where \(A^c\) is the complement of \(A\).
Definition 2.2  [10] A subset $A$ of a topological space $(X, \tau)$ is called a $\tilde{g}$-closed set if $\text{cl} (A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open.

Definition 2.3  [9] A subset $A$ of a topological space $(X, \tau)$ is called a $\tilde{g}^s$-closed set if $\text{cl} (A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tilde{g}$-open.

Note that M.K.R.S. Veerakumar called it ’$g$-closed in his 2006 paper[9].

Definition 2.4  A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is continuous if $f^{-1} (U)$ is closed in $X$ for each closed set $U$ in $Y$.

Definition 2.5  [1] A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is $g$-continuous if $f^{-1} (U)$ is $g$-closed in $X$ for each closed set $U$ in $Y$.

Definition 2.6  [5] A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is irresolute if $f^{-1} (U)$ is semi closed in $X$ for each semi closed set $U$ in $Y$.

Definition 2.7  [10] A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is $\tilde{g}$-irresolute if $f^{-1} (U)$ is $\tilde{g}$-closed in $X$ for each $\tilde{g}$-closed set $U$ in $Y$.

Definition 2.8  [5] A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is pre semi closed if $f (V)$ is semi closed in $Y$ for each semi closed set $V$ in $X$.

Definition 2.9  [2] A topological space $(X, \tau)$ is called a $T_0$ space if every $gs$-closed set is closed.

Definition 2.10  [2] A topological space $(X, \tau)$ is called a $T_0$ space if every $gs$-closed set is $g$-closed.

Definition 2.11  [6] A topological space $(X, \tau)$ is called a $T_{1/2}$ space if every $g$-closed set is closed in $X$.

Lemma 2.12  If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is irresolute,

then for every subset $B$ of $Y$, $\text{cl} \left( f^{-1} (B) \right) \subseteq f^{-1} \left( \text{cl} (B) \right)$.
Proof. Let \( x \in sdl\left(f^{-1}(B)\right) \).

Suppose that \( V \) is any semi open set of \( Y \) containing \( f(x) \).

i.e., \( f(x) \in V \).

Then \( x \in f^{-1}(V) \).

Since \( f \) is irresolute, \( f^{-1}(V) \) is semi open set of \( X \) and \( f^{-1}(V) \cap f^{-1}(B) \neq \phi \).

\[ \Rightarrow f^{-1}(V \cap B) \neq \phi. \]

\[ \Rightarrow (V \cap B) \neq \phi \]

\[ \Rightarrow f(x) \in scl(B). \]

\[ \Rightarrow x \in f^{-1}(f(x)) \subseteq f^{-1}(sdl(B)) \]

\[ \Rightarrow sdl\left(f^{-1}(B)\right) \subseteq f^{-1}(scl(B)). \]

\( \blacksquare \)

Notations used:

(i) \( \hat{g}^sC(X, \tau) \) denotes the class of all \( \hat{g}^s \)-closed sets in \( (X, \tau) \).

(ii) \( \hat{g}^sO(X, \tau) \) denotes the class of all \( \hat{g}^s \)-open sets in \( (X, \tau) \).

(iii) \( scl(A) \) denotes semi closure of \( A \)

(iv) \( scl(A) \) denotes semi interior of \( A \)

3 \( \hat{g}^s \)-continuous functions

Definition 3.1 \[8\] A subset \( A \) of a topological space \( (X, \tau) \) is called a \( \hat{g}^s \)-closed set if \( scl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \hat{g} \)-open.

Definition 3.2 A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \hat{g}^s \)-continuous if \( f^{-1}(U) \) is \( \hat{g}^s \)-closed for each closed set \( U \) in \( Y \).

Definition 3.3 A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \hat{g}^s \)-irresolute if \( f^{-1}(U) \) is \( \hat{g}^s \)-closed for each \( \hat{g}^s \)-closed set \( U \) in \( Y \).

Example 3.4 Let \( (X, \tau) \) and \( (Y, \sigma) \) be two topological spaces where \( X = Y = \{a, b, c, d\} \) with \( \tau = \{\phi, X, \{a\}, \{a, b, c\}, \{a, d\}\} \) and \( \sigma = \{\phi, X, \{a\}, \{a, b\}, \{a, b, c\}\} \).

Then \( \tau^c = \{\phi, X, \{b, c, d\}, \{d\}, \{b, c\}\} \) and \( \sigma^c = \{\phi, X, \{b, c, d\}, \{c, d\}, \{d\}\} \).

Also \( \hat{g}^sC(X, \tau) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{d\}, \{c\}, \{b\}\} \).
Define \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = b, f(b) = d, f(c) = a, f(d) = c \).

We have \( f^{-1}([b, c, d]) = \{a, b, d\}, f^{-1}([c, d]) = \{b, d\}, f^{-1}([d]) = \{b\} \).

Thus \( f^{-1}(U) \) is \( \hat{g}^*s \)-closed for each closed set \( U \) in \( Y \).

Therefore \( f \) is \( \hat{g}^*s \)-continuous.

**Proposition 3.5** The following are equivalent for \( f : (X, \tau) \rightarrow (Y, \sigma) \).

(i) \( f \) is \( \hat{g}^*s \)-continuous.

(ii) \( f^{-1}(U) \) is \( \hat{g}^*s \)-open for each open set \( U \) in \( Y \).

**Proof.** (i) \( \Rightarrow \) (ii)

Suppose that \( f \) is \( \hat{g}^*s \)-continuous. Let \( U \) be open in \( Y \). Then \( U^c \) is closed in \( Y \). Since \( f \) is \( \hat{g}^*s \)-continuous, we have \( f^{-1}(U^c) \) is \( \hat{g}^*s \)-closed in \( X \). But \( f^{-1}(U^c) = [f^{-1}(U)]^c \). Therefore \( f^{-1}(U) \) is \( \hat{g}^*s \)-open in \( X \).

(ii) \( \Rightarrow \) (i)

Suppose that \( f^{-1}(U) \) is \( \hat{g}^*s \)-open for each open set \( U \) in \( Y \). Let \( V \) be closed in \( Y \). Then \( V^c \) is open in \( Y \). By assumption, \( f^{-1}(V^c) \) is \( \hat{g}^*s \)-open in \( X \). i.e., \( f^{-1}(V) \) is \( \hat{g}^*s \)-closed in \( X \). Thus \( f \) is \( \hat{g}^*s \)-continuous. \( \blacksquare \)

**Proposition 3.6** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function.

(i) \( f \) is \( \hat{g}^*s \)-continuous.

(ii) For each \( x \) in \( X \) and for each open set \( V \) containing \( f(x) \), there is a \( \hat{g}^*s \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \).

(iii) \( f(\hat{g}^*scl(A)) \subseteq cl(f(A)) \) for each subset \( A \) of \( X \).

(iv) \( \hat{g}^*scl(f^{-1}(B)) \subseteq f^{-1}(cl(f(B))) \) for each subset \( B \) of \( Y \).

Then (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv)

**Proof.** (i) \( \Rightarrow \) (ii)

Let \( x \in X \) and \( V \) be an open set containing \( f(x) \). Then, by (i), \( f^{-1}(V) \) is a \( \hat{g}^*s \)-open set of \( X \) containing \( x \). If \( U = f^{-1}(V) \), then \( f(U) = f(f^{-1}(V)) \subseteq V \).

(ii) \( \Rightarrow \) (iii)

Let \( A \) be a subset of a space \( X \) and \( f(x) \notin cl(f(A)) \). Then there exists open set \( V \) of
\(Y\) containing \(f(x)\) such that \(V \cap f(A) = \phi\). Now, by \((ii)\), there is a \(\hat{g}^s\)-open set \(U\) containing \(x\) such that \(f(x) \in f(U) \subseteq V\). Hence \(f(U) \cap f(A) = \phi\) i.e., \(f(U \cap A) = \phi\) i.e., \(U \cap A = \phi\). Therefore \(x \notin \hat{g}^s \text{cl}(A)\). Therefore \(f(x) \notin f(\hat{g}^s \text{cl}(A))\). Therefore \(f(\hat{g}^s \text{cl}(A)) \subseteq \text{cl}(f(A))\).

\((iii) \Rightarrow (iv)\)

Let \(B\) be a subset of \(Y\) such that \(A = f^{-1}(B)\). By \((iii)\), \(f(\hat{g}^s \text{cl}(A)) \subseteq \text{cl}(f(A)) \subseteq \text{cl}(B)\). Therefore \(\hat{g}^s \text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))\).

**Lemma 3.7**  \[8\] A subset \(A\) of a topological space \((X, \tau)\) is \(\hat{g}^s\)-open iff \(F \subseteq \text{sint}(A)\) whenever \(F \subseteq A\) and \(F\) is \(\hat{g}\)-closed.

**Proposition 3.8**  Let \(B\) be a \(\hat{g}^s\)-open (or \(\hat{g}^s\)-closed) subset of \((Y, \sigma)\) satisfying \(\text{sint}(B) = \text{int}(B)\). Then \(f^{-1}(B)\) is \(\hat{g}^s\)-open (or \(\hat{g}^s\)-closed) in \((X, \tau)\) if \(f: (X, \tau) \longrightarrow (Y, \sigma)\) is \(\hat{g}^s\)-continuous and if image of a \(\hat{g}\)-closed set in \(X\) under \(f\) is \(\hat{g}\)-closed set in \(Y\).

**Proof.** Let \(B\) be a \(\hat{g}^s\)-open set in \(Y\). Let \(F \subseteq f^{-1}(B)\) where \(F\) is a \(\hat{g}\)-closed set in \(X\). Then \(f(F) \subseteq B\) holds. By our assumption, \(f(F)\) is \(\hat{g}\)-closed set in \(Y\) and \(B\) be a \(\hat{g}^s\)-open in \(Y\). Therefore, by Lemma 3.7, \(f(F) \subseteq \text{sint}(B)\) holds. Again, by our assumption, \(f(F) \subseteq \text{int}(B)\) and hence \(F \subseteq f^{-1}(\text{int}(B))\) holds. Since \(f\) is \(\hat{g}^s\)-continuous and \(\text{int}B\) is open in \(Y\), \(f^{-1}(\text{int}(B))\) is \(\hat{g}^s\)-open in \(X\). So, by Lemma 3.7, \(F \subseteq \text{sint}(f^{-1}(\text{int}(B)))\) holds i.e., \(F \subseteq \text{sint}(f^{-1}(\text{int}(B))) \subseteq \text{sint}(f^{-1}(B))\) holds. Therefore \(f^{-1}(B)\) is \(\hat{g}^s\)-open. By taking complements, we can show that if \(B\) is \(\hat{g}^s\)-closed in \(Y\), then \(f^{-1}(B)\) is \(\hat{g}^s\)-closed in \(X\).

**Proposition 3.9**  The following are equivalent for \(f: (X, \tau) \longrightarrow (Y, \sigma)\).

\((i)\) \(f\) is \(\hat{g}^s\)-irresolute.

\((ii)\) \(f^{-1}(U)\) is \(\hat{g}^s\)-open for each \(\hat{g}^s\)-open set \(U\) in \(Y\).

**Proof.** \((i) \Rightarrow (ii)\)

Suppose that \(f\) is \(\hat{g}^s\)-irresolute. Let \(U\) be \(\hat{g}^s\)-open in \(Y\). Then \(U^c\) is \(\hat{g}^s\)-closed in \(Y\). Since \(f\) is \(\hat{g}^s\)-irresolute, we have \(f^{-1}(U^c)\) is \(\hat{g}^s\)-closed in \(X\). But \(f^{-1}(U^c) = (f^{-1}(U))^c\). Therefore \(f^{-1}(U)\) is \(\hat{g}^s\)-open in \(X\).
(ii) ⇒ (i)

Suppose that $f^{-1}(U)$ is $\hat{g}^*s$-open for each $\hat{g}^*s$-open set $U$ in $Y$. Let $V$ be $\hat{g}^*s$-closed in $Y$. Then $V^c$ is $\hat{g}^*s$-open in $Y$. Therefore $f^{-1}(V^c)$ is $\hat{g}^*s$-open in $X$. Therefore $f^{-1}(V)$ is $\hat{g}^*s$-closed in $X$. Therefore $f$ is $\hat{g}^*s$-irresolute.

**Proposition 3.10** If a function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is $\hat{g}^*s$-irresolute, then $f$ is $\hat{g}^*s$-continuous.

**Proof.** Let $V$ be a closed set of $Y$. But every closed set is $\hat{g}^*s$-closed. Therefore $V$ is a $\hat{g}^*s$-closed set of $Y$. Since $f$ is $\hat{g}^*s$-irresolute, $f^{-1}(V)$ is $\hat{g}^*s$-closed in $X$. Therefore, by Definition 3.2, $f$ is $\hat{g}^*s$-continuous.

**Remark 3.11** The converse of Proposition 3.10 need not be true as seen from the following example.

**Example 3.12** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces where $X = Y = \{a, b, c, d\}$ with $\tau = \{\phi, X, \{a\}, \{a, b, c\}\}$ and $\sigma = \{\phi, X, \{a\}, \{a, b\}, \{a, b, c\}\}$. Then $\tau^c = \{\phi, X, \{b, c, d\}, \{d\}\}$ and $\sigma^c = \{\phi, X, \{b, c, d\}, \{c, d\}, \{d\}\}$.

$\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{a, d\}, \{d\}, \{c\}, \{b\}\}.$

$\hat{g}^*sC(Y, \sigma) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{a, d\}, \{d\}, \{c\}, \{b\}\}.$

Define $f : X \rightarrow Y$ by $f(a) = a, f(b) = c, f(c) = d, f(d) = b$.

We have $f^{-1}(\{b, c, d\}) = \{a, b, c\}, f^{-1}(\{c, d\}) = \{a, c, d\}, f^{-1}(\{d\}) = \{c\}$. Thus $f^{-1}(U)$ is $\hat{g}^*s$-closed for each closed set $U$ in $Y$.

Therefore $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\hat{g}^*s$-continuous.

But, $f^{-1}(\{a, c, d\}) = \{a, b, c\}$ is not $\hat{g}^*s$-closed in $X$, whereas $\{a, c, d\}$ is $\hat{g}^*s$-closed in $Y$.

Therefore $f : (X, \tau) \rightarrow (Y, \sigma)$ is not $\hat{g}^*s$-irresolute.

**Proposition 3.13** Let $Y$ be a $T_b$ space. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\hat{g}^*s$-irresolute if it is $\hat{g}^*s$-continuous.

**Proof.** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be $\hat{g}^*s$-continuous. Let $A$ be a $\hat{g}^*s$-closed set in $Y$. But
every $g^s$-closed set is $gs$-closed. Therefore $A$ is $gs$-closed in $Y$. Since $Y$ is a $T_b$ space, $A$ is closed. Since $f$ is $g^s$-continuous, $f^{-1}(A)$ is $g^s$-closed in $X$. Hence $f$ is $g^s$-irresolute.

**Proposition 3.14** Let $Y$ be a $T_d$-space and $T_{1/2}$-space. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $g^s$-irresolute if it is $g^s$-continuous.

**Proof.** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be $g^s$-continuous. Let $A$ be a $g^s$-closed set in $Y$. But every $g^s$-closed set is $gs$-closed. Therefore $A$ is $gs$-closed in $Y$. Since $Y$ is a $T_d$-space, $A$ is $g$-closed in $Y$. Since $Y$ is a $T_{1/2}$-space, $A$ is closed in $Y$. Since $f$ is $g^s$-continuous, $f^{-1}(A)$ is $g^s$-closed in $X$. Therefore $f$ is $g^s$-irresolute in $X$.

**Proposition 3.15** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \mu)$ be two functions. Let $Y$ be a $T_{1/2}$-space, $g$ a $g$-continuous function and $f$ a $g^s$-continuous function. Then $g \circ f$ is $g^s$-continuous.

**Proof.** Let $U$ be closed in $Z$. Since $g$ is $g$-continuous, $g^{-1}(U)$ is $g$-closed in $Y$. But $Y$ is $T_{1/2}$. Therefore $g^{-1}(U)$ is closed in $Y$. Since $f$ is $g^s$-continuous, $f^{-1}(g^{-1}(U))$ is $g^s$-closed. Therefore $g \circ f$ is $g^s$-continuous.

**Proposition 3.16** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be $g^s$-irresolute and $X$ is $T_b$. Then $f$ is continuous.

**Proof.** Let $V$ be a closed subset of $Y$. Then $V$ is semi closed and hence $g^s$-semi-irresolute in $Y$. Since $f$ is $g^s$-irresolute, $f^{-1}(V)$ is $g^s$-closed in $X$. But every $g^s$-closed is $gs$-closed. Therefore $f^{-1}(V)$ is $gs$-closed in $X$. But $X$ is $T_b$. Therefore $f^{-1}(V)$ is closed in $X$. Therefore $f$ is continuous.

**Proposition 3.17** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be $g^s$-irresolute and $X$ is $T_b$. Then $f$ is irresolute.
Proof. Let $V$ be a semi closed subset of $Y$. Then $V$ is $\hat{g}^s$-closed in $Y$. Since $f$ is $\hat{g}^s$-irresolute, $f^{-1}(V)$ is $\hat{g}^s$-closed in $X$. But every $\hat{g}^s$-closed is $g$-closed. Therefore $f^{-1}(V)$ is $g$-closed in $X$. But $X$ is $T_h$. Therefore $f^{-1}(V)$ is closed in $X$. Then $f^{-1}(V)$ is semi closed in $X$ and hence $f$ is irresolute.

Lemma 3.18 If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is surjective and if image of a $\hat{g}$-closed set is $\hat{g}$-closed under $f$, then for every subset $S$ of $Y$ and each $\hat{g}$-open set $U$ of $X$ containing $f^{-1}(S)$, there exists $\hat{g}$-open set $V$ of $Y$ such that $S \subseteq V$ and $(f^{-1}(V)) \subseteq U$.

Proof. Let $S \subseteq Y$ and $U$ be a $\hat{g}$-open set in $X$, containing $f^{-1}(S)$.

Put $V = Y - f(X - U)$.

Then $V$ is $\hat{g}$-open in $Y$ containing $S$.

$\Rightarrow f^{-1}(V) = f^{-1}(Y - f(X - U))$.

$\Rightarrow f^{-1}(V) = X - f^{-1}((f(X - U)))$.

$\Rightarrow f^{-1}(V) \subseteq X - (X - U)$.

$\Rightarrow f^{-1}(V) \subseteq U$.

Proposition 3.19 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be surjective and if image of a $\hat{g}$-closed set is $\hat{g}$-closed under $f$. Then for every $\hat{g}^s$-closed set $B$ in $Y$, $f^{-1}(B)$ is $\hat{g}^s$-closed in $X$.

Proof. Let $B$ be a $\hat{g}^s$-closed set in $Y$. Suppose that $f^{-1}(B) \subseteq U$ where $U$ is $\hat{g}$-open set of $X$. By assumption and by Lemma 3.18, there is $\hat{g}$-open set $V$ in $Y$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$. Since $B$ is $\hat{g}^s$-closed in $Y$ and $B \subseteq V$, we have $scl(B) \subseteq V$. Hence $f^{-1}(scl(B)) \subseteq f^{-1}(V) \subseteq U$. By Lemma 2.12, $sd(f^{-1}(B)) \subseteq U$. Therefore $f^{-1}(B)$ is $\hat{g}^s$-closed in $X$.

Proposition 3.20 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function such that $f$ is pre semi closed and $\hat{g}$-irresolute. Then for every $\hat{g}^s$-closed set $A$ in $X$, $f(A)$ is $\hat{g}^s$-closed set in $Y$.

Proof. Let $A$ be a $\hat{g}^s$-closed set in $X$. Suppose that $f(A) \subseteq U$ where $U$ is $\hat{g}$-open set in $Y$. Then $A \subseteq f^{-1}(U)$ and $f^{-1}(U)$ is $\hat{g}$-open set in $X$. Since $A$ is $\hat{g}^s$-closed in $X$, $scl(A) \subseteq f^{-1}(U)$ and hence $f(scl(A)) \subseteq U$. But $sd(f(A)) \subseteq scl(f(scl(A)))$. Since $f$
is pre semi closed, \( scl (f(A)) \subseteq f(scl(A)) \). Therefore \( scl (f(A)) \subseteq U \). Hence \( f(A) \) is \( \hat{g}^s \)-closed set in \( Y \).

**Proposition 3.21** If a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \hat{g}^s \)-irresolute, then for every subset \( A \) of \( X \), \( f(\hat{g}^s sd (A)) \subseteq scl (f(A)) \).

**Proof.** Let \( A \subseteq X \). We know that every semi closed set is \( \hat{g}^s \)-closed. Therefore, we have \( scl (f(A)) = \hat{g}^s \)-closed in \( Y \). Since \( f \) is \( \hat{g}^s \)-irresolute, \( f^{-1}(scl (f(A))) \) is \( \hat{g}^s \)-closed in \( X \). Also \( A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(scl (f(A))) \). Since \( f^{-1}(scl (f(A))) \) is \( \hat{g}^s \)-closed, \( \hat{g}^s sd (A) \subseteq f^{-1}(scl (f(A))) \). Therefore \( f(\hat{g}^s scl(A)) \subseteq f\{f^{-1}(scl (f(A)))\} \subseteq scl (f(A)) \). ■

**Proposition 3.22** If a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is bijective, \( \hat{g}^s \)-continuous, \( scl(A) = cl (A) \) for all subsets \( A \) in \( Y \) and if image of a \( \hat{g} \)-open set is \( \hat{g} \)-open under \( f \), then \( f \) is \( \hat{g}^s \)-irresolute.

**Proof.** Let \( V \) be a \( \hat{g}^s \)-closed set of \( Y \). Let \( f^{-1}(V) \subseteq U \) where \( U \) is \( \hat{g} \)-open in \( X \). Then \( f(f^{-1}(V)) \subseteq f(U) \). Since \( f \) is surjective, \( V \subseteq f(U) \). Since \( f(U) \) is \( \hat{g} \)-open and since \( V \) is \( \hat{g}^s \)-closed in \( Y \), we have \( scl(V) \subseteq f(U) \). By our assumption, \( d(V) \subseteq f(U) \). Since \( f \) is injective, \( f^{-1}(d(V)) \subseteq U \). Since \( f \) is \( \hat{g}^s \)-continuous and since \( d(V) \) is closed in \( Y \), \( f^{-1}(d(V)) \) is \( \hat{g}^s \)-closed in \( X \). Therefore \( scl(f^{-1}(d(V))) \subseteq U \). Since \( V \subseteq d(V) \), we have \( scl(f^{-1}(V)) \subseteq U \). Therefore \( f^{-1}(V) \) is \( \hat{g}^s \)-closed in \( X \) and hence \( f \) is \( \hat{g}^s \)-irresolute. ■

**Definition 3.23** A map \( f : X \rightarrow Y \) is called a \( \hat{g}^s \)-closed map if \( f(U) \) is \( \hat{g}^s \)-closed in \( (Y, \sigma) \) for every closed set \( U \) of \( (X, \tau) \).

**Definition 3.24** A map \( f : X \rightarrow Y \) is called a \( \hat{g}^s \)-open map if \( f(U) \) is \( \hat{g}^s \)-open in \( (Y, \sigma) \) for every open set \( U \) of \( (X, \tau) \).

**Proposition 3.25** If \( f : X \rightarrow Y \) is \( \hat{g} \)-irresolute and \( \hat{g}^s \)-closed and \( A \) is a \( \hat{g}^s \)-closed subset of \( X \), then \( f(A) \) is \( \hat{g}^s \)-closed in \( Y \).

**Proof.** Let \( f(A) \subseteq U \) and \( U \) is \( \hat{g} \)-open in \( Y \). Then \( f^{-1}(f(A)) \subseteq f^{-1}(U) \). i.e., \( A \subseteq f^{-1}(U) \). Since \( f \) is \( \hat{g} \)-irresolute, \( f^{-1}(U) \) is \( \hat{g} \)-open in \( X \). Since \( A \) is \( \hat{g}^s \)-closed, \( cl(A) \subseteq f^{-1}(U) \). So,
\( f(\text{cl}(A)) \subseteq f(\text{cl}(U)). \) i.e., \( f(\text{cl}(A)) \subseteq U. \) Since \( f \) is \( \hat{g}^s \)-closed and \( \text{cl}(A) \) is closed in \( X, f(\text{cl}(A)) \) is \( \hat{g}^s \)-closed in \( Y. \) Therefore \( \text{scl}(f(\text{cl}(A))) \subseteq U. \) Since \( f(A) \subseteq f(\text{cl}(A)), \) we have \( \text{scl}(f(A)) \subseteq \text{scl}(f(\text{cl}(A))) \subseteq U. \) Therefore \( f(A) \) is \( \hat{g}^s \)-closed in \( Y. \)

**Proposition 3.26** If \( f : X \rightarrow Y \) is \( \hat{g}^s \)-closed and \( g : Y \rightarrow Z \) is \( \hat{g} \)-irresolute and \( \hat{g}^s \)-closed, then \( g \circ f : X \rightarrow Z \) is \( \hat{g}^s \)-closed.

**Proof.** Let \( F \) be a closed set of \( X. \) Since \( f \) is \( \hat{g}^s \)-closed, \( f(F) \) is \( \hat{g}^s \)-closed in \( Y. \) Since \( g \) is \( \hat{g} \)-irresolute and \( \hat{g}^s \)-closed and \( f(F) \) is \( \hat{g}^s \)-closed in \( Y, \) by proposition 3.25, \( g(f(F)) \) is \( \hat{g}^s \)-closed in \( Z. \) Hence \( g \circ f : X \rightarrow Z \) is \( \hat{g}^s \)-closed.

**Proposition 3.27** If \( f : X \rightarrow Y \) is closed and \( g : Y \rightarrow Z \) is \( \hat{g}^s \)-closed, then \( g \circ f : X \rightarrow Z \) is \( \hat{g}^s \)-closed.

**Proof.** Let \( F \) be a closed set of \( X. \) Since \( f \) is closed, \( f(F) \) is closed in \( Y. \) Since \( g \) is \( \hat{g}^s \)-closed, \( g(f(F)) = g \circ f(F) \) is \( \hat{g}^s \)-closed in \( Z. \) Hence \( g \circ f : X \rightarrow Z \) is \( \hat{g}^s \)-closed.

**Proposition 3.28** Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be two maps such that \( g \circ f : X \rightarrow Z \) be a \( \hat{g}^s \)-open map. Then \( g \) is \( \hat{g}^s \)-open, if \( f \) is continuous and surjective.

**Proof.** Let \( A \) be open in \( Y. \) Since \( f \) is continuous, \( f^{-1}(A) \) is open in \( X. \) Since \( f^{-1}(A) \) is open in \( X, g \circ f(f^{-1}(A)) \) is \( \hat{g}^s \)-open in \( Z. \) i.e., \( g(A) \) is \( \hat{g}^s \)-open in \( Z. \) Therefore, \( g \) is a \( \hat{g}^s \)-open map.

**Proposition 3.29** For any bijection \( f : X \rightarrow Y, \) the following are equivalent:

(i) \( f^{-1} : Y \rightarrow X \) is \( \hat{g}^s \)-continuous.

(ii) \( f \) is \( \hat{g}^s \)-open.

(iii) \( f \) is \( \hat{g}^s \)-closed.

**Proof.** (i) \( \Rightarrow \) (ii)

Let \( F \) be open in \( X. \) Then \( X - F \) is closed in \( X. \) Since \( f^{-1} \) is \( \hat{g}^s \)-continuous, \( (f^{-1})^{-1}(X - F) = f(X - F) = Y - f(F) \) is \( \hat{g}^s \)-closed in \( Y. \) Then \( f(F) \) is \( \hat{g}^s \)-open in \( Y. \) Hence \( f \) is \( \hat{g}^s \)-open.
(ii) ⇒ (iii)

Let \( F \) be closed in \( X \). Then \( X - F \) is open in \( X \). Since \( f \) is \( \hat{g}^s \)-open, \( f(X - F) = Y - f(F) \) is \( \hat{g}^s \)-open in \( Y \). Then \( f(F) \) is \( \hat{g}^s \)-closed in \( Y \). Hence \( f \) is \( \hat{g}^s \)-closed.

(iii) ⇒ (i)

Let \( V \) be closed in \( X \). Since \( f : X \rightarrow Y \) is \( \hat{g}^s \)-closed, \( f(V) \) is \( \hat{g}^s \)-closed in \( Y \).

i.e., \( f^{-1}(V) \) is \( \hat{g}^s \)-closed in \( Y \). Therefore \( f^{-1} \) is \( \hat{g}^s \)-continuous.

References


[8] S. Poius Missier and M. Anto, \( \hat{g}^s \)-closed sets in topological spaces, (submitted)