PERIODIC HARVESTING OF RENEWABLE RESOURCES SUBJECTED TO ADDITIVE ALLEE EFFECT

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Abstract. Allee effect refers to a reduction in individual fitness at low population density that can lead to extinction [1] Allee effect occurs whenever fitness of an individual in a small or sparse population decreases as the population size or density also declines [1], [2]. Bio-economics of renewable resource in seasonally varying environment has been presented in [3]. In this article we analyze a logistic model with Allee effect in proportionate periodic harvesting also we study the bio-economics of a renewable resources subject to additive Allee effect. The Pontryagin maximum principle has been employed to solve the considered optimal harvesting problem [4].

KEYWORDS: Allee effect, Additive Allee effect, stable solution, positive solution.

1. INTRODUCTION

Population models with harvesting

In this section we investigate the nature of solutions of population growth model subject to harvesting term in a seasonally varying environment. We assume that extra mortality rate and cooperative factor of the considered population are periodic of the same period T. Thus we consider the following models:

\[ \frac{dy}{dt} = y\left(1 - y - \frac{\eta(t)}{1 + m(t)y}\right) - E(t)y \]  

with

\[ y(0) = y_0 > 0. \]

For biological reasons we will be interested in non-negative solution of Eq. (1.1). In this study \( \eta(t) \) and \( m(t) \) represents the extra mortality rate and cooperative factor respectively.

Lemma 1: For a periodic harvest \( E(t) \) (of period T), if \( 1 - E(t) \) is negative then solution of Eq. (1.1) eventually approaches the zero solution \( y(t) = 0 \).

Proof. Eq. (1.1) can be written as

\[ \frac{dy}{dt} = y\left(1 - E(t) - \frac{\eta(t)}{1 + m(t)y}\right) \]

If \( (1 - E(t)) < 0 \) then \( \frac{dy}{dt} < 0 \) this implies that \( \lim_{n \to \infty} y(t)\left(1 + \frac{1}{m(t)}\right)^n \).

Hence the proof is complete.

Lemma 2: For a periodic harvest effort \( E(t) \) (of periodic T), if \( 1 - E(t) \) is positive then Eq. (1.1) admits a positive nontrivial periodic solution of period T.

Proof. Eq. (1.3) can be written as

\[ \frac{dy}{dt} = -(1 - E(t)y) = -f(t, y) \]

where \( f(t, y) = y^2 + \frac{\eta(t)}{1 + m(t)y} \). Solving Eq. (1.4) using variation of constant formula we have
\[ y(t) = y(0) e^{\int_0^t (1 - E(s)) \, ds} - e^{\int_0^t (1 - E(s)) \, ds} \int_0^T e^{\int_0^u (1 - E(u)) \, du} f(s, y(s)) \, ds \]

\[ = \frac{\int_0^T e^{\int_0^u (1 - E(u)) \, du} f(s, y(s)) \, ds}{1 - e^{-\int_0^T (1 - E(s)) \, ds}}. \]

Since \( f(t, y) \) and \( 1 - e^{\int_0^T (1 - E(u)) \, du} \) are positive implies that the term \( \int_0^T e^{\int_0^u (1 - E(u)) \, du} f(s, y(s)) \, ds \) is positive. Also \( 1 - E(t) \) is positive implies that \( 1 - e^{-\int_0^T (1 - E(s)) \, ds} \) is positive.

Hence \( y(t) \) is positive.

### 2. OPTIMAL HARVESTING POLICY IN SEASONALLY VARYING ENVIRONMENT

In this section we wish to obtain an optimal harvesting policy \( E(t) \) which maximizes the time stream of net revenues \( I(E) \), to the infinite horizon when the resource dynamics is governed by the equation considered in the previous section. Thus we have the following optimal control problem.

\[ \max_E I(E) = \int_0^\infty e^{-\delta t} \left( pyE(t) - cE(t) \right) dt \]

subject to the dynamic constraint

\[ \frac{dy}{dt} = y \left( 1 - y - \frac{\eta(t)}{1 + m \eta(t) y} \right) - E(t) y \]

With \( y(0) = y_0, E(t) \in [0, E_{max}] \).

Where \( \delta \) is the instantaneous annual rate of discount, \( p \) is the price per unit harvest, \( c \) is the cost per unit effort and \( E_{max}(E_{min}) \) represents the maximum (minimum) allowable effort.

The current value of the Hamiltonian is

\[ H = pyE(t) - cE(t) + \lambda \left[ f(t, y) - E(t)y \right]. \]

The dynamics of the costate variable \( \lambda \) (shadow) price for \( \lambda \) is given by \( \dot{\lambda} = \delta \lambda - \frac{\partial H}{\partial y} \) which is equivalent to \( \dot{\lambda} = \delta \lambda - (pE(t) + \lambda f_y - \lambda E(t)) \)

\[ \frac{\partial H}{\partial E} = py - c - \lambda y \text{ and from } \frac{\partial H}{\partial y} = 0 \text{ we have } \lambda(t) = p - \frac{c}{y}. \]

From Eq. (2.2) and Eq. (2.6) we have

\[ \frac{d\lambda}{dt} = \frac{c}{y^2} \left( 1 - y - \frac{\eta(t)}{1 + m \eta(t) y} \right) \]

From Eq. (2.5), Eq. (2.6) and Eq. (2.7) we have

\[ \frac{c}{y} \left( 1 - y - \frac{\eta(t)}{1 + m \eta(t) y} \right) - \frac{c}{y} \left( \delta \left( p - \frac{c}{y} \right) - pE(t) - (p - \frac{c}{y}) f_y + \left( p - \frac{c}{y} \right) E(t) \right) \]

Rearranging and simplifying Eq. (2.8) we obtain

\[ y^4 + \frac{1}{2} \left( \delta - 1 - \frac{c}{p} + \frac{4}{m(t)} \right) y^3 + \frac{1}{2} \left( \frac{2c}{m(t)} - \frac{2c}{m(t)} \right) y^2 + \frac{1}{2} \left( \frac{c}{m^2(t)} - \frac{c}{pm(t)} + \frac{c}{pm(t)} \right) y - \frac{c^3}{2pm^2(t)} = 0 \]

The singular trajectory of the population is given by Eq. (2.9) which is nothing but the positive solution, denoted by \( y_s(t), t > 0 \). Using the Eq. (2.2) we obtain the singular effort policy to be

\[ E_s(t) = 1 - y_s(t) = \frac{\eta(t)}{1 + m \eta(t) y_s(t)} - \frac{dy}{y_s(t)} \]

Where

\[ E_s(t) = 1 - y_s(t) = \frac{\eta(t)}{1 + m \eta(t) y_s(t)} + \frac{dy}{y_s(t)} \]

\[ = 1 - y_s(t) = \frac{\eta(t)}{1 + m \eta(t) y_s(t)} + \frac{N}{D} \]

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\[ N = 4p m(t) m'(t) y^3_s - [2p m(t) m'(t) - 2\delta p m(t) m'(t) + 2c m(t) m'(t) - 4p m'(t)] y^2_s \]
\[ + [2p m'(t) - 2\delta p m'(t) + 2c m'(t) + 2c\delta m(t) m'(t)] y_s \]
\[ + [p \eta'(t) + c m'(t) \eta(t) + cm(t) \eta'(t) - 2c \delta m'(t)]. \]

and
\[ D = - 8p m^2(t) y^3_s + 3[pm^2(t) - 4p m(t) + c m(t)] y^2_s \]
\[ + 2[2p m(t) - 2\delta p m(t) + 2c m(t) - 2p + \delta c m^2(t)] y_s \]
\[ + p + c - p\eta(t) - cm(t) \eta'(t) - \delta p + 2c \delta c m(t). \]

We emphasize that \( E_s(t) \) is a function of \( t \) only. It can be easily verified that the singular effort policy reduces to the policy developed by Clark [5] in this case the coefficients \( m(t) \) and \( \eta(t) \) are constants and \( \eta(t) = 0. \)

**Lemma 3**: The function \( y_s(t) \) and \( E_s(t) \) are periodic of the same period \( T. \)

Proof. First we will show that \( y_s(t) \) is well defined. Observe that, for each fixed \( t \geq 0, \) denoted by \( \tilde{t} \) Eq. (2.9) defines a bi-quadratic equation in \( y(\tilde{t}) \) with negative term without \( y \) and the discriminate may be positive or negative. Thus Eq. (2.9) admits at least one positive root for each fixed \( \tilde{t} \). Hence \( y_s(t) \) is well defined for all \( t \geq 0. \) Since all the coefficients of the Eq. (2.9) are periodic functions of period \( T, \) the periodicity of \( y_s(t) \) follows. Now, to prove that \( E_s(t) \) is well defined it is sufficient to show that the term represented by \( D \) is different from zero for each fixed \( \tilde{t} \). Suppose, by the way of contradiction, that the above term is equal to zero for some \( \tilde{t}. \) This would imply, recognizing the fact that the expression \( D \) is the derivative of Eq. (2.9) with respect to \( y, \) that \( y_s(\tilde{t}) \) is the double zero of Eq. (2.9) at \( t = \tilde{t}, \) which is a contradiction due to the fact that the constant term of Eq. (2.9) is negative for all \( \tilde{t}. \) Thus \( E_s(t) \) is well defined and its periodicity follows from its definition.

Let \( E^*(t) \) represents the harvest policy constructed from the singular effort policy \( E_s(t), \) which is restriction of amplitude of \( E_s(t) \) to \([E_{min}, E_{max}]\) and let the corresponding asymptotic periodic solution of Eq. (1.1) with \( E(t) = E^*(t) \) be denoted by \( y^*(t) \) which is termed as optimal solution. Note that \( E^*(t) \) will in general be a combination of singular and bang-bang controls defined as follows

\[ E^*(t) = \begin{cases} 
E_s(t), & \text{if } E_{min} \leq E_s(t) \leq E_{max} \\
E_{min}, & \text{if } E_s(t) \leq E_{min} \\
E_{max}, & \text{if } E_s(t) \geq E_{max}.
\end{cases} \]

Clearly, \( E^*(t) \) is also a periodic function of period \( T. \) Now we need to show to reach the solution \( y^*(t) \) optimally from a given initial state \( y(0). \) In view of statements (1), (2) and (3), the solution \( y^*(t) \) is globally asymptotically stable and hence this solution can be reached by applying the bang-bang control policy initially as follows

\[ \tilde{E}(t) = \begin{cases} 
E_{max}, & \text{if } y(0) > y_s(0) \text{ and } y(t) > y^*(t) \\
E_{min}, & \text{if } y(0) < y_s(0) \text{ and } y(t) < y^*(t) \\
E_s(t), & \text{if } y(t) = y^*(t).
\end{cases} \]

If \( E^*(t) \) is equal to the singular harvest policy \( E_s(t) \) then the above bang-bang control policy reduces to the well known policy given in terms of the switching function as follows

\[ \tilde{E}(t) = \begin{cases} 
E_{max}, & \text{if } \sigma(t) > 0 \\
0, & \text{if } \sigma(t) < 0.
\end{cases} \]

Where \( \sigma(t) = py - c - \lambda y = (p - \lambda) y - c \) is the switching function.

Let \( \tau \) be the minimum time at which the path \( y(t) \) generated by the bang-bang policy in the interval \([0, \tau] \) is termed as optimal approach path. Then the optimal harvest policy is given by

\[ E^0(t) = \begin{cases} 
\tilde{E}(t), & \text{if } 0 \leq t \leq \tau \\
E^*(t) if \ t > \tau.
\end{cases} \]
and the optimal path is given by the trajectory generated by the above control. In view of the global asymptotic stability of the solution $y^*(t)$, it also possible to reach $y^*(t)$ using a sub optimal harvest policy given by

$$E_{so} = E^*(t) \text{ for } t \geq 0.$$ 

The path traced by the state under this sub optimal harvest policy is termed as sub optimal path. Clearly advantage is choosing the optimal harvest policy is that the state reaches the optimal solution $y^*(t)$ in a finite time while in the case of the said suboptimal harvest policy it reaches the optimal solution asymptotically.

3. REFERENCES