Fixed Point Theorems for Cyclic Contractions with C-Class Functions on B-Metric Spaces
Clement Ampadu (Corresponding Author) and Arslan Hojat Ansari

Abstract: Let $F$ be a $C - class$ function [A. H. Ansari, Note on $\varphi - \psi$ -contractive type mappings and related fixed point, The 2\textsuperscript{nd} Regional Conference on Mathematics and Applications, PNU, September 2014, pages 377-380] and let $\psi$ and $\phi$ be altering distance functions [M.S. Khan, M. Swaleh, S. Sessa, Fixed point theorems by altering distances between the points, Bull. Aust. Math. Soc. 30 (1984), 12-18]. In this paper we introduce a concept of cyclic generalized $F(\psi, \phi)$-rational contraction and use it to prove the existence and uniqueness of fixed points of this type of contraction mapping in b-metric spaces. We apply our results to obtain fixed points of certain contractions of integral type.

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I. Introduction and Preliminaries

Definition 1 [I.A. Bakhtin, The contraction mapping principle in quasimetric spaces, Funct. Anal., Unianowsk Gos. Ped.Inst. 30 (1989), 26-37]: Let $X$ be a nonempty set. The mapping $d: X \times X \to [0, \infty)$ is called a b-metric if it satisfies the following:

(a) $d(x, y) = 0$ iff $x = y$ for all $x, y \in X$
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$
(c) there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$

Example 2 [H.R. Roshan et.al, Fixed Point Theory Appl. 2013 (2013), Article ID 159]: Let $(X, d)$ be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number, then $\rho(x, y) = (d(x, y))^p$ is a b-metric with $s = 2^{p-1}$

Remark 3: The definition of convergence and Cauchy sequence in b-metric spaces as well as the definition of completeness in b-metric spaces is defined similarly as in the usual metric space. For these definitions, see for example [I.A. Bakhtin, The contraction mapping principle in quasimetric spaces, Funct. Anal., Unianowsk Gos. Ped.Inst. 30 (1989), 26-37].

Definition 4 [W.A. Kirk, P.S. Srinivasan, P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory, 4 (2003), 79-89]: Let $(X, d)$ be a metric space, $p$ be a positive integer, $A_i$ be nonempty subsets of $X$ for $i = 1,2, \ldots , p$, $Y = \bigcup_{i=1}^{p} A_i$, and $T: Y \to Y$. We say $T: Y \to Y$ is a cyclic operator if

(a) $A_i$ are nonempty subsets of $X$ for $i = 1,2, \ldots , p$
(b) $T(A_p) \subseteq T(A_1)$ and $T(A_{i-1}) \subseteq T(A_i)$ for $i = 2,3, \ldots , p$

Remark 5: For definition and examples of C-class functions see [A. H. Ansari, Note on $\varphi - \psi$ -contractive type mappings and related fixed point, The 2\textsuperscript{nd} Regional Conference on Mathematics and Applications, PNU, September 2014, pages 377-380]
Remark 6: For definition of altering distance function see Khan et.al [M.S. Khan, M. Swaleh, S. Sessa, Fixed point theorems by altering distances between the points, Bull. Aust. Math. Soc. 30 (1984), 12-18]

Definition 7: Let \((X, d)\) be a complete b-metric space, \(p\) be a positive integer, \(A_i\) be nonempty closed subsets of \(X\) for \(i = 1,2, \ldots, p\), and \(Y = \bigcup_{i=1}^{p} A_i\). An operator \(T: Y \to Y\) will be called a cyclic generalized \(F(\psi, \phi)\)-rational contraction if

\[(a) \ T: Y \to Y\ is a cyclic operator\]
\[(b) \ For\ any \ x \in A_i, y \in A_{i+1}, i = 1,2, \ldots, p, \ \psi \ and \ \phi \ being\ altering\ distance\ functions, \ L \geq 0, \ A_{p+1} = A_1 \ and \ F \ is \ a \ C - class\ function, \ one \ has, \]
\[\psi(s^4d(Tx,Ty)) \leq F \left(\psi(M(x, y)), \phi(M(x, y))\right) + L\psi(m(x, y)), \text{where}\]
\[M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}^{\frac{1+d(x,Tx)}{1+d(x,y)}}, d(x,Ty)+d(y,Tx) + \frac{d(x,Tx)+d(y,Ty)}{2s}\], and
\[m(x, y) = \min\{d(x, Tx) + d(y, Ty), d(x, Ty), d(y, Tx)\}\]

II. Main Results

Theorem 8: Let \((X, d)\) be a complete b-metric space, \(p\) be a positive integer, \(A_i\) be nonempty closed subsets of \(X\) for \(i = 1,2, \ldots, p\), and \(Y = \bigcup_{i=1}^{p} A_i\). Suppose that \(T: Y \to Y\) is a cyclic generalized \(F(\psi, \phi)\)-rational contraction, then \(T: Y \to Y\) has a unique fixed point in \(\bigcap_{i=1}^{p} A_i\).

If \(s = 1, p = 1 \ and \ A_1 = X\) in the above theorem then we get the following

Corollary 9: Let \((X, d)\) be a complete b-metric space and \(T: X \to X\) be such that for all \(x, y \in X\) and \(L \geq 0\), one has,
\[\psi(d(Tx,Ty)) \leq F \left(\psi(M(x, y)), \phi(M(x, y))\right) + L\psi(m(x, y)), \text{where}\]
\[M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}^{\frac{1+d(x,Tx)}{1+d(x,y)}}, d(x,Ty)+d(y,Tx) + \frac{d(x,Tx)+d(y,Ty)}{2}\], and
\[m(x, y) = \min\{d(x, Tx) + d(y, Ty), d(x, Ty), d(y, Tx)\}\]

Then \(T: X \to X\) has a unique fixed point.

III. Applications

Let \(\Delta\) denote the set of functions \(f: [0, \infty) \to [0, \infty)\) such that

\[a) \ f: [0, \infty) \to [0, \infty) \ is \ Lebesgue\ integrable\ on\ each\ compact\ subset\ of\ [0, \infty)\]
\[b) \ \int_{0}^{\infty} f(t) dt > 0 \ for\ each\ \epsilon > 0\]

We easily notice that \(\psi: [0, \infty) \to [0, \infty)\) defined by \(\psi(t) = \int_{0}^{\infty} f(t) dt\) is an altering distance function. Thus the following are immediate results

Corollary 10: Let \((X, d)\) be a complete b-metric space, \(p\) be a positive integer, \(A_i\) be nonempty closed subsets of \(X\) for \(i = 1,2, \ldots, p\), and \(Y = \bigcup_{i=1}^{p} A_i\). Suppose that \(T: Y \to Y\) is a mapping satisfying the following conditions:
\[\int_{0}^{s^4d(Tx,Ty)} f(t) dt \leq F \left(\int_{0}^{M(x,y)} f(t) dt, \int_{0}^{M(x,y)} g(t) dt\right) + L\int_{0}^{m(x,y)} f(t) dt, \text{for any} \ x \in A_i, y \in A_{i+1}, \]
\(i = 1,2, \ldots, p, \ \psi \ and \ \phi \ being\ altering\ distance\ functions, \ L \geq 0, \ A_{p+1} = A_1 \ and \ F \ is \ a \ C - class\ function, \ where\]

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\( M(x, y) = \max\{d(x, y), d(x, T_x), d(y, T_y)\frac{1+d(x, T_x)}{1+d(x, y)}\frac{d(x, T_y)+d(y, T_x)}{2s}\}, \) and \( m(x, y) = \min\{d(x, T_x) + d(y, T_y), d(x, T_y), d(y, T_x)\}\), then \( T: Y \to Y \) has a unique fixed point.

**Remark 11:** If \( s = 1, p = 1 \) and \( A_1 = X \) in Corollary 10, then we get the integral version of Corollary 9.

## IV. Concluding Remarks

In the present paper we have shown the usefulness of \( C - class \) functions in the proof of fixed point theorems. In particular their usefulness in given new interpretation of certain contractive definitions in the literature. Note that \( F(s, t) = s - t \) is a \( C - class \) function, therefore we have the following relationships with existing results in the literature

- a) Definition 3.7 in R.George et.al [J. Nonlinear Funct. Anal. 2015, 2015:5, ISSN: 2052-532X] can be recovered from Definition 7 above by taking, \( F \left( \psi(M(x, y)), \phi(M(x, y)) \right) = \psi(M(x, y)) - \phi(M(x, y)) \)

- b) Theorem 3.8 in R.George et.al [J. Nonlinear Funct. Anal. 2015, 2015:5, ISSN: 2052-532X] can be recovered from Theorem 8 above by taking \( F \left( \psi(M(x, y)), \phi(M(x, y)) \right) = \psi(M(x, y)) - \phi(M(x, y)) \)

- c) Corollary 3.9 in R.George et.al [J. Nonlinear Funct. Anal. 2015, 2015:5, ISSN: 2052-532X] can be recovered from Corollary 9 above by taking \( F \left( \psi(M(x, y)), \phi(M(x, y)) \right) = \psi(M(x, y)) - \phi(M(x, y)) \)

Note that if \( f(t) = 1 = g(t) \) in Corollary 10, and \( \psi \) and \( \phi \) are the identity functions in Theorem 8, then Corollary 10 and Theorem 8 coincide. Alternatively, one could prove Corollary 10 from Theorem 8 by taking \( \psi(u) = \int_0^u f(t) dt \) and \( \phi(u) = \int_0^u g(t) dt \).

## References


