DISTRIBUTIONAL FINITE-GENERALIZED-LAPLACE-HANKEL-CLIFFORD-TRANSFORMATION

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Abstract: The distributional finite-generalized-Laplace-Hankel-Clifford transforms is defined and inversion theorem is established in distributional sense. Operational transform formula is obtained for developed finite-generalized-Laplace-Hankel-Clifford transformation. These are applied to solve certain partial differential equations with distributional boundary conditions.

Keywords: finite generalized Hankel-Clifford transforms, Operational transform formula, distributions, testing spaces.

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1. INTRODUCTION


\[ (h_{1,\alpha,\beta}f)(n) = F_{1,\alpha,\beta}(n) = \int_0^a x^{-\alpha-\beta} j_{\alpha,\beta}(j_n x) f(x) dx, \] (1.1)

which is called finite generalized Hankel-Clifford transformation of first kind of order \((\alpha - \beta)\). Its inversion theorem is stated as

Theorem 1.1. Let \(f(t)\) be a function defined in \((0,1)\) and assumed to be absolutely summable over the same interval. Let \((\alpha - \beta) \geq -\frac{1}{2}\) and

\[ a_n = \frac{1}{\mathcal{J}_{\alpha,\beta-1}(j_n)} \int t^{-\alpha-\beta} j_{\alpha,\beta}(j_n t) f(t) dt, n = 1, 2, ... \]

If \(f(t)\) is of bounded variation in \((a,b)\), \((0 < a < b < 1)\) and if \(x \in (a,b)\), then the series (1.2) converges to \(\frac{1}{2} [f(x + 0) + f(x - 0)]\).

Earlier Malgonde studied generalized Hankel-Clifford transformation of certain spaces of distributions in [3] and developed generalized Hankel-Clifford transformation of arbitrary order in [4]. Later Malgonde and Lakshmi Gorty [9], extended the finite generalized Hankel-Clifford transformation to a class of generalized functions, which gives rise to the Fourier-Bessel series expansion of the generalized function; the convergence of the series is interpreted in the weak distributional sense. The finite Hankel-Clifford transformation to a certain space of generalized functions by employing kernel method was established, the techniques employed was different from those employed in previous works.
It is quite well known that there are several problems which can be solved by the repeated applications of the transformations and in particular the transformations in (1.1) and (1.3). If constructed an integral transform for which the kernel is the product of the kernels of the Laplace and Hankel-Clifford transformation of the first kind, integral (special) transform as Laplace Hankel-Clifford transform which has been successfully applied to deal with the problems occurring in mathematical physics analogous to [2].

The finite-generalized-Laplace-Hankel-Clifford transform is defined by

$$L_{\alpha,\beta}(f)(s, \lambda_m) = F(s, \lambda_m) = \int_{-\infty}^{\infty} e^{-sx} y^{-\alpha-\beta} J_{\alpha,\beta} (\lambda_m y) f(x, y) \, dx \, dy,$$

where \( f(x, y) \) belongs to an appropriate function space \((-\infty < x < \infty, 0 < y < 1)\), and where

$$J_{\alpha,\beta}(y) = y^{(\alpha+\beta)/2} J_{\alpha-\beta} (2\sqrt{y})$$

being the Bessel function of the first kind of order \((\alpha - \beta)\) with \((\alpha - \beta) \geq -\frac{1}{2}\), \(\Re[\alpha] > -1 \) & \(\Re[s \log e] > 0\) and \(s = \sigma + iR\) is a restricted complex variable. The finite-generalized-Laplace-Hankel-Clifford transform is represented as FGLHCT in [11].

2. PRELIMINARY RESULTS

Theorem 2.1 (The Analyticity theorem)

If \( L_{\alpha,\beta}(f)(s, \lambda_m) = F(s, \lambda_m) \) for \( s \in \Delta_f = \{s \mid \sigma_f < \Re(s) < \rho_f\} \) and \( \{\lambda_m\} \) are the positive zeros of \( J_{\alpha,\beta}(z) \), \( F(s, \lambda_m) \) is analytic in \( s \) for some fixed \( \lambda_m \) and \( D[k] F(s, \lambda_m) = \{f(x, y), -xe^{-sx} y^{-\alpha-\beta} J_{\alpha,\beta} (\lambda_m y)\}\).

Theorem 2.2 Let \( f \) be a member \( LH'_{a,b,\alpha,\beta} \), \( \alpha - \beta \geq -\frac{1}{2}, \alpha \leq \Re(s) \leq b, m = 1, 2, 3, \ldots \) and \( F(s, \lambda_m) \) be defined by \( F(s, \lambda_m) = \{f(x, y), e^{-sx} y^{-\alpha-\beta} J_{\alpha,\beta} (\lambda_m y)\} \).

The operators defined in [5] are:

\[
\Delta_{\alpha,\beta,y} = D_y^2 + \frac{(1-\alpha-\beta)}{y} D_y + \frac{\alpha \beta}{y^2},
\]

\[
D_k^k \Delta_{\alpha,\beta,y} = D_k \left( D_y^2 + \frac{(1-\alpha-\beta)}{y} D_y + \frac{\alpha \beta}{y^2} \right)^k, k, k' = 0, 1, 2, 3, \ldots
\]

where \((\alpha - \beta) \geq -\frac{1}{2}\) and the expression, \(T_N(y, \tau) = \sum_{m=1}^n \frac{\rho_{\alpha,\beta} (\lambda_m y) \rho_{\alpha,\beta} (\lambda_m \tau)}{\lambda_m^2 \rho_{\alpha,\beta} (\lambda_m)}\).

3. The SPACES \( LH_{a,b,\alpha,\beta}, LH(w, z, \alpha, \beta) \) and their duals.

Let \( a, b, \alpha - \beta \) be real numbers such that; \(\alpha - \beta \geq -\frac{1}{2}\), and let \( k^{(s)} \) be the function defined as

\[k^{(s)}(x) = \begin{cases} e^{-ax} & -\infty < x < \infty.
\end{cases}\]

Then \( LH_{a,b,\alpha,\beta} \) is defined as the linear space of all complex valued smooth functions \( \phi(x, y) \) on \((-\infty < x < \infty, 0 < y < 1)\) such that for each \( k, k' = 0, 1, 2, 3, \ldots\)

\[
\rho_{a,\alpha,\beta}(k, k' \mid \phi(x, y)) = \sup_{-\infty < x < \infty, 0 < y < 1} k^{(s)}(x) y^b D_k^k \Delta_{\alpha,\beta,y} \left[ y^{a+b} \phi(x, y) \right] < \infty.
\]

Assign to \( LH_{a,b,\alpha,\beta} \) the topology generated by the semi-norms \( \rho_{a,\alpha,\beta}(k, k', 0) \).

Hence \( LH_{a,b,\alpha,\beta} \) is countablymultinormed space which is complete. The dual space \( LH'_{a,b,\alpha,\beta} \) consists of all continuous linear functionalson \( LH_{a,b,\alpha,\beta} \). By the [10, P.21] \( LH'_{a,b,\alpha,\beta} \) is also complete. If \( a \leq d \) and \( e \leq b \),
then $LH_{d,e,a,b} \subseteq LH_{a,b,a,b}$ and the topology of $LH_{d,e,a,b}$ is stronger than the topology induced on it by $LH_{a,b,a,b}$. Consequently the restriction of any member $f \in LH'_{a,b,a,b}$ to $LH_{d,e,a,b}$ is in $LH'_{d,e,a,b}$.

A certain countable union space $LH(w,z,\alpha,\beta)$ is considered. Let $w$ denote either a finite real number or $-\infty$ and $z$ and a denote either a finite real number or $+\infty$. Choose two monotonic sequences $\{a_v\}_{v=1}^\infty$ and $\{b_v\}_{v=1}^\infty$ such that $a_v \to w$ and $b_v \to z$. Then $LH(w,z,\alpha,\beta)$ is defined as countable union space of all $LU_{a_v,b_v,\alpha,\beta}$ spaces; thus $LH(w,z,\alpha,\beta)$ is complete and hence a countable-union space, $LH(w,z,\alpha,\beta)$ is complete. $LH'(w,z,\alpha,\beta)$ denotes the dual space of $LH(w,z,\alpha,\beta)$. Hence $LH'(w,z,\alpha,\beta)$ is also complete [10, p. 25].

Now several facts to which will be referred later:

i) Clearly, $D(\Omega)$ is sub-space of $LH_{a,b,a,b}$ as well as of $LH(w,z,\alpha,\beta)$ whatever be the value of $a, w$ or $z$; the convergence in $D(\Omega)$ implies the convergence in $LH_{a,b,a,b}$ and also convergence in $LH(w,z,\alpha,\beta)$.

ii) Consequently, the restriction of any member of $LH'_{a,b,a,b}$ or $LH'(w,z,\alpha,\beta)$ to $D(\Omega)$ is a member of $D'(\Omega)$. Hence the member of $LH'_{a,b,a,b}$ and $LH'(w,z,\alpha,\beta)$ are distributions in the sense of Zemanian [10, p. 39].

iii) Since $D(I)$ is dense in $LH(w,z,\alpha,\beta)$ for every $w, z$ therefore by Theorem 1.9.1 [10, p. 24] $LH'(w,z,\alpha,\beta)$ is a subspace of $D'(\Omega)$.

iv) Let $w \leq x$ and $y \leq z$, then $LH(x,y,\alpha,\beta) \subseteq LH(w,z,\alpha,\beta)$ and convergence in $LH(x,y,\alpha,\beta)$ implies the convergence in $LH(w,z,\alpha,\beta)$. Since $D(\Omega) \subseteq LH(x,y,\alpha,\beta)$ and $D(\Omega)$ is dense in $LH(w,z,\alpha,\beta)$, $LH(x,y,\alpha,\beta)$ is also dense in $LH(w,z,\alpha,\beta)$. Hence by Theorem 1.9.1[10, p. 24] $LH'(w,z,\alpha,\beta)$ is a subspace of $LH'(x,y,\alpha,\beta)$.

v) If $f(x,y)$ is locally integrable function defined on $-\infty < x < \infty, 0 < y < 1$ and if

$$\int_{-\infty}^{\infty} \int_{0}^{1} \left[k_a(x)\right]^{-1} y^{\alpha+\beta-1} f(x,y) \, dx \, dy$$

exists, then $f(x,y)$ generates a regular generalized function on $LH'_{a,b,a,b}$ through the definition $\langle f, \phi \rangle = \int_{-\infty}^{\infty} \int_{0}^{1} f(x,y) \phi(x,y) \, dx \, dy; \phi \in LH_{a,b,a,b}$.

Similarly, if $w < a$ and $b < z$, then $f$ generates a regular member of $LH'(w,z,\alpha,\beta)$ through the definition

$$\langle f, \phi \rangle = \int_{-\infty}^{\infty} \int_{0}^{1} f(x,y) \phi(x,y) \, dx \, dy; \phi \in LH(w,z,c,\alpha,\beta).$$

For each $m = 1, 2, 3, ...$ and $\alpha - \beta \geq -1/2, b \geq 1/2$ and $a \leq \text{Re}(s) \leq b$ the function $e^{-sx} y^{\alpha-\beta} \int_{a,b}(\lambda_m y)$ is a member of $LH_{a,b,a,b}$. For all $w < a$ and $b < z$, $e^{-sx} y^{\alpha-\beta} \int_{a,b}(\lambda_m y)$ is a member of $LH(w,z,\alpha,\beta)$.

4.INVERSION AND UNIQUENESS.

An inversion formula for FGLHCT. The proof of the inversion formula requires some lemmas.

**Lemma 4.1** Let $LH_{a,b}(f) = F(s,\lambda_m)$ for $s \in \Omega$, and $\lambda_m$, let $\phi(x,y) \in D(\Omega)$, and set for $0 < a' < b' < 1$

$$\varphi(s,\lambda_m) = \int_{-\infty}^{\infty} \int_{a'}^{b'} y^{\alpha-\beta} e^{-sx} \int_{a,b}(\lambda_m y) \phi(x,y) \, dx \, dy.$$  

Then for any fixed real number $R, 0 < R < \infty$
\[ R \int_{-R}^{R} \left( f(t, \tau), e^{-it \tau} J_{a, \beta}(\lambda_m \tau) \phi(s, \lambda_m) dw = \left\{ f(t, \tau), \int_{-R}^{R} e^{-it \tau} J_{a, \beta}(\lambda_m \tau) \phi(s, \lambda_m) dw \right\} \]  

(4.1)

where \( s = \sigma + iw \) and \( \sigma \) is fixed with \( \sigma_f < \sigma < \rho_f \).

**Proof.** In the case of \( \phi(x, y) = 0 \), the proof is obvious.

Now considering \( \phi(x, y) \neq 0 \). Then first to show that \( v(t, r) = \int_{-R}^{R} e^{-itr} J_{a, \beta}(\lambda_m r) \phi(s, \lambda_m) dw \) is a member of \( LH_{a, b, a, \beta}(I) \), for some real number \( a \) and \( b \) such that \( a < \sigma_f < \sigma < \rho_f < b \). To prove this for all \( k, k' = 0, 1, 2, 3, \ldots \)

\[ \sup_{(t, r) \in I} \left| k_a(t) r^b D^k \Lambda^{k'}_{a, \beta, r} \left[ r^{a+\beta} \int_{-R}^{R} e^{-it \tau} J_{a, \beta}(\lambda_m \tau) \phi(s, \lambda_m) dw \right] \right| \]

is finite. Because of smoothness of the integrand, taking the operator inside the integral sign.

Hence

\[ D^k \Lambda^{k'}_{a, \beta, r} \left[ r^{a+\beta} \int_{-R}^{R} e^{-it \tau} J_{a, \beta}(\lambda_m \tau) \phi(s, \lambda_m) dw \right] = (-1)^{k+k'} \lambda_m^{2k'} J_{a, \beta}(\lambda_m r) \int_{-R}^{R} e^{-st} s^k \phi(s, \lambda_m) dw. \]

Hence,

\[ \sup_{(t, r) \in I} \left| k_a(t) r^b D^k \Lambda^{k'}_{a, \beta, r} \left[ r^{a+\beta} \int_{-R}^{R} e^{-it \tau} J_{a, \beta}(\lambda_m \tau) \phi(s, \lambda_m) dw \right] \right| \]

\[ = \sup_{(t, r) \in I} \left| k_a(t) r^{b+a+\beta} (-1)^{k+k'} \lambda_m^{2k'+a+\beta} J_{a, \beta}(\lambda_m r) \int_{-R}^{R} e^{-st} s^k \phi(s, \lambda_m) dw \right| \]

\[ \leq A_{a, \beta} \lambda_m^{2k'+a+\beta} r^{b+a+\beta} k_a(t) e^{-\sigma t} \int_{-R}^{R} s^k \phi(s, \lambda_m) dw \]

\[ < \infty \]

for \( k_a(t) e^{-\sigma t} \) is bounded in \( -\infty < t < \infty; a \leq \sigma \leq b \) and \( A_{a, \beta} \) is a bound for \( (\lambda_m r)^{a-\beta} J_{a, \beta}(\lambda_m r) \), where \( 0 < r < 1 \).

This ensures that \( \int_{-R}^{R} e^{-itr} J_{a, \beta}(\lambda_m r) \phi(s, \lambda_m) dw \) is a member of \( LH_{a, b, a, \beta}(I) \). As it is obvious that the function \( e^{-itr} J_{a, \beta}(\lambda_m r) \) is a member of the space \( LH_{a, b, a, \beta}(I) \) with \( t, r \) as the variables of testing functions, the expressions on both sides have sense of (4.1).

The Riemann-sum is given by

\[ \theta_M(t, r) = \sum_{i=1}^{M} e^{-is_i t} r^{-a-\beta} J_{a, \beta}(s_i \lambda_m) \frac{2R}{M} \]

(4.2)

where \( s_i = \sigma + iw \). Since \( f(t, r) \) is linear, then by applying \( f(t, r) \) to (4.2) term by term we get

\[ \left\{ f(t, r), \theta_M(t, r) \right\} = \sum_{i=1}^{M} \left\{ f(t, r), e^{-is_i t} r^{-a-\beta} J_{a, \beta}(s_i \lambda_m) \phi(s_i, \lambda_m) \right\} \frac{2R}{M} \]

\[ \rightarrow \int_{-R}^{R} \left\{ f(t, r), e^{-it \tau} J_{a, \beta}(\lambda_m \tau) \phi(s, \lambda_m) dw \right\} \]

by virtue of continuity of the integrand on \( -R < w < R \).
Since \( f \in LH'_{a,b,\alpha,\beta}(I) \), if shown that \( \theta_M(t, r) \) converges in \( LH_{a,b,\alpha,\beta}(I) \) to
\[
\int_{-R}^{R} e^{-\alpha r - \beta} \mathcal{P}_{a,\alpha,\beta}(\lambda_m r) \phi(s, \lambda_m) \, dw = M \rightarrow \infty, \text{ for } M \rightarrow \infty, \text{ proof completes. For this purpose, to prove}
\]
\[
P_{a,\alpha,\beta}^b \left[ \theta_M(t, r) - v(t, r) \right] \rightarrow 0 \text{ uniformly on } -\infty < t < \infty; 0 < r < 1 \text{ as } M \rightarrow \infty.
\]
Considering
\[
A_M(t, r) = k_\alpha(t) r^\beta \Delta_{t, \alpha, \beta, r}^M \left[ r^{\alpha + \beta} \left[ \theta_M(t, r) - v(t, r) \right] \right]
\]
(4.3)
\[
A_M(t, r) = k_\alpha(t) r^\beta \left[ \sum_{i=1}^{M} (-1)^{i+k'} e^{-\alpha r - \beta} s^i \lambda_m^{2k'} \mathcal{P}_{a,\alpha,\beta}(\lambda_m r) \phi(s, \lambda_m) \frac{2R}{M} \right]
\]
(4.4)
Now
\[
\left| k_\alpha(t) e^{-\alpha r - \beta} \mathcal{P}_{a,\alpha,\beta}(\lambda_m r) \right| \rightarrow 0 \text{ as } |t| \rightarrow \infty, 0 < r < 1.
\]
So, for given \( \varepsilon > 0 \), choose \( T \) so large that for all \( |t| > T \).
\[
\left| k_\alpha(t) e^{-\alpha r - \beta} \mathcal{P}_{a,\alpha,\beta}(\lambda_m r) \right| < \frac{\varepsilon}{2} \left[ \lambda_m^{2k'+\alpha+\beta} r^{b+\alpha+\beta} \int_{-R}^{R} s^k \phi(s, \lambda_m) \, dw \right]^{-1}.
\]
Hence for all \( |t| > T \), the magnitude of second term of right hand side of (4.4) is bounded by \( \frac{\varepsilon}{2} \). Again for \( |t| > T \), the magnitude of first term on right hand side of (4.4) is bounded by
\[
\frac{1}{2} \left[ \lambda_m^{2k'+\alpha+\beta} r^{b+\alpha+\beta} \int_{-R}^{R} s^k \phi(s, \lambda_m) \, dw \right]^{-1} \left[ \lambda_m^{2k'+\alpha+\beta} r^{b+\alpha+\beta} \sum_{i=1}^{M} s^i \phi(s, \lambda_m) \frac{2R}{M} \right].
\]
Now choosing \( M \) so large,that \( M > M_0 \), the last expression is less than \( \frac{\varepsilon}{2} \). Therefore for all \( |t| > T \) and \( M > M_0 \), \( A_M(t, r) \) < \( \varepsilon \).
Finally, \( k_\alpha(t) e^{-\alpha r - \beta} \mathcal{P}_{a,\alpha,\beta}^{b+\alpha+\beta}(\lambda_m r) \phi(s, \lambda_m) \) is a uniformly continuous function on the domain \( \{-T < t < T, -R < w < R, 0 < r < 1\} \). Therefore in view of (4.4), there exists \( M_1 \) such that for all \( M \geq M_1 \), \( A_M(t, r) < \varepsilon \) on \( -T < t < T, 0 < r < 1 \). Thuswhen \( M_2 = \max(M_0, M_1) \), \( A_M(t, r) < \varepsilon \) on \( -\infty < t < \infty, 0 < r < 1 \) for all \( M \geq M_2 \). Hence \( A_M(t, r) \) converges uniformly to zero on \( -\infty < t < \infty, 0 < r < 1 \) as \( M \rightarrow \infty \).

**Lemma 4.2.** Let \( a', b' \) be any two real numbers satisfying \( 0 < a' < b' < 1 \). Then
\[
\lim_{R, N \to \infty} \int_{-\infty}^{\infty} \int_{a}^{b'} \frac{y^{-\alpha-\beta} T_N(y, \tau)}{(x-t)} \sin \frac{R(x-t)}{(x-t)} \, dxdy = \pi
\]
(4.5)
when \( -\infty < t < \infty, a' < \tau < b' \).

**Proof.** It is a fact that for \( R > 0 \),
\[
\int_{-\infty}^{\infty} \frac{\sin R(x-t)}{(x-t)} \, dx = \int_{-\infty}^{\infty} \frac{\sin Rx}{x} \, dx = \pi
\]
Hence,
\[
\lim_{R,N \to \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^{-a-b}T_N(y,\tau) \frac{\sin R(x-t)}{(x-t)} \, dx\, dy
\]
\[
= \lim_{N \to \infty} \int_{-\infty}^{\infty} y^{-a-b}T_N(y,\tau) \left[ \lim_{K \to \infty} \frac{\sin R(x-t)}{(x-t)} \right] \, dy
\]
\[
= \pi \lim_{N \to \infty} \int_{-\infty}^{\infty} y^{-a-b}T_N(y,\tau) \, dy
\]
\[
= \pi.
\]

**Lemma 4.3.** Let \( a, b \) and \( R \) such that \( a < \sigma < b \) and let \( \psi(x,y) \in D(\Omega) \). Then for fixed \( y, \psi(x,y) \in D(\Omega) ; \quad \Omega = \{(x,y)\mid -\infty < x < \infty, y \text{ is fixed}\} \)

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \psi(x,y)e^{\sigma(x-y)} \frac{\sin R(x-t)}{(x-t)} \, dx
\]

(4.6)

converges in \( L^1_h(a,b,a,b)(t) \) to \( \psi(t,y) \) as \( R \to \infty \)

**Proof.** In the following assume that \( R > 0 \). It is a fact that \( \int_{-\infty}^{\infty} \frac{\sin R(x-t)}{(x-t)} \, dx = \int_{-\infty}^{\infty} \frac{\sin Rx}{x} \, dx = \pi \) as \( R \to \infty \). Thus objective is to prove that for each \( k,k' = 0,1,2,3,... \)

\[
\theta_k(t,y) = \frac{1}{\pi} k_a(t) y^{b-a-\beta} D_k^k \left\{ \int_{-\infty}^{\infty} \left[ \psi(x,y) e^{\sigma(x-y)} - \psi(t,y) \right] \frac{\sin R(x-t)}{x-t} \, dx \right. \\
- \left. \int_{-\infty}^{\infty} \left[ \psi(x+y,t) e^{\sigma x} - \psi(t,y) \right] \frac{\sin Rx}{x} \, dx \right\}
\]

converges uniformly to zero on the domain \( \{(t,y)\mid -\infty < t < \infty, y \text{ is fixed}\} \). Here \( D_t = \frac{\partial}{\partial t} \).

Since \( \psi(x,y) \) is smooth and of bounded support so that for fixed \( y \), \( \phi(x,y) \) is also smooth and is bounded support, differentiating under integral sign:

\[
\theta_k(t,y) = \frac{1}{\pi} k_a(t) y^{b-a-\beta} \int_{-\infty}^{\infty} \left\{ \left( e^{\sigma x} D_t^k \psi(x+t,y) - D_t^k \psi(t,y) \right) \frac{\sin Rx}{x} \, dx \right\}
\]

(4.7)

\[
= I_{1,R}(t,y) + I_{2,R}(t,y) + I_{3,R}(t,y)
\]

Here \( I_{1,R}(t,y), I_{2,R}(t,y) \) and \( I_{3,R}(t,y) \) denote the quantities obtained by interchanging over the intervals \( -\infty < x < -\delta, -\delta < x < \delta \) and \( \delta < x < \infty \) respectively and \( y \) is fixed; where \( \delta > 0 \).

First consider \( I_{2,R}(t,y) \). The function \( H(x,t,y) = k_a(t) y^{b-a-\beta} x^{-a-\beta} \left( e^{\sigma x} D_t^k \psi(x+t,y) - D_t^k \psi(t,y) \right) \)

(4.8)

is continuous function of \( (x,t,y) \) for all \( t > 0; x \neq 0 \) and \( y \) is fixed. Since \( \psi \) is smooth (4.8) tends to \( k_a(t) y^{b-a-\beta} D_t^k \psi(x+t,y) \)

(4.9)

as \( x \to 0 \). Upon assigning the value of \( H(0,t,y) \) to (4.9), function \( H(x,t,y) \) is continuous everywhere on the \( (x,t,y) \) plane. Since \( \psi \) is of bounded support, \( H(x,t,y) \) is bounded on the domain \( \{(x,t,y)\mid -\delta < x < \delta, -\infty < t < \infty, y \text{ is fixed}\} \) by, say the constant \( M \). Thus \( \varepsilon > 0 \), choice of \( \delta \) should be so small that
\[ |I_{2,R}(t, y)| = \left| \frac{1}{\pi} \int_{-\delta}^{\delta} H(x, t, y) \frac{\sin Rx}{x} \, dx \right| \]
\[ \leq \frac{2M \delta}{\pi} > \epsilon \]

\(-\infty < t < \infty, y \) is fixed. Set \( I_{1,R}(t, y) = J_{1,\alpha,\beta,R}(t, y) - J_{2,\alpha,\beta,R}(t, y) \), where

\[ J_{1,\alpha,\beta,R}(t, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} k_{a,b}(t) y^{b-a-\beta} e^{\sigma x} D_1^b \psi(x+t, y) \frac{\sin Rx}{x} \, dx \]

\[ J_{2,\alpha,\beta,R}(t, y) = \frac{1}{\pi} k_{a,b}(t) y^{b-a-\beta} D_1^b \psi(t, y) \int_{-\infty}^{\infty} \frac{\sin z}{z} \, dz. \]

Since \( k_{a}(t) y^{b-a-\beta} D_1^b \psi(t, y) \) is continuous and of bounded support, it is bounded on \( \{(t, y) \mid -\infty < t < \infty, y \text{ is fixed} \} \). By the convergence of improper integral \( \int_{-\infty}^{\infty} \frac{\sin z}{z} \, dz \), it follows therefore that \( J_{2,\alpha,\beta,R}(t, y) \) tends uniformly to zero on \( -\infty < t < \infty, y \text{ is fixed} \) as \( R \to \infty \). To show that \( J_{1,\alpha,\beta,R}(t, y) \) does the same, first integrate by parts and use the fact \( \psi(t, y) \) is of bounded support to obtain

\[ J_{1,\alpha,\beta,R}(t, y) = \frac{1}{\pi} e^{-\alpha \delta} k_{a}(t) y^{b-a-\beta} D_1^b \psi(x+t, y) \frac{\cos R \delta}{\delta} + \frac{1}{\pi} \int_{-\infty}^{\infty} \cos (Rx) k_{a}(t) y^{b-a-\beta} D_x \left[ D_1^b \psi(x+t, y) \frac{e^{\alpha x}}{x} \right] \, dx \]

The first term on right hand side of (4.11) tends to zero as \( R \to \infty \), since \( \sigma, \delta, y \) are fixed and \( k_{a}(t) y^{b-a-\beta} D_1^b \psi(t-\delta, y) \) is bounded function of \( (t, y) \). Moreover

\[ k_{a}(t) y^{b-a-\beta} D_x \left[ \frac{e^{\alpha x}}{x} D_1^b \psi(x+t, y) \right] = k_{a}(t) y^{b-a-\beta} \left[ \frac{\sigma}{x} - \frac{1}{x^2} \right] e^{\alpha x} \left[ D_1^b \psi(x+t, y) \right] \]

\[ + k_{a}(t) y^{b-a-\beta} \frac{e^{\alpha x}}{x} D_x \left[ D_1^b \psi(x+t, y) \right] \]

But, for every \( k, k_{a}(t) y^{b-a-\beta} e^{\alpha x} D_1^b \left[ \psi(x+t, y) \right] \) is bounded on \( (x, y, t) \) plane. This is because \( D_1^b \left[ \psi(x+t, y) \right] \) is bounded and has its support contained in the strip \( \{(x, y, t) \mid |x+t| < A \} \) where as \( e^{\alpha x} k_{a}(t) \) is also bounded on this strip by virtue of inequality \( a < \sigma < b \). Thus, (4.11) is bounded on the domain \( \{(x, y, t) \mid -\delta < x < \delta, -\infty < t < \infty, y \text{ is fixed} \} \) by, say the constant \( N \).

This result and the assumption that the support of \( \psi(t, y) \) is contained in \( [A, B] \times [a', b'] \) imply that second term on right hand side of (4.11) is bounded by \( \frac{2NA}{\pi R} \) which tends to zero as \( R \to \infty \). So,

\( J_{1,\alpha,\beta,R}(t, y) \) and therefore \( I_{1,R}(t, y) \to 0 \) uniformly on \( \{(t, y) \mid -\infty < t < \infty, y \text{ is fixed} \} \) as \( R \to \infty \). Similarly we can prove \( I_{3,R}(t, y) \) also tends uniformly to zero on \( \{(t, y) \mid -\infty < t < \infty, y \text{ is fixed} \} \) as \( R \to \infty \). Established that \( \lim_{R \to \infty} \left| \theta_R(t, y) \right| \leq \epsilon \) and since \( \epsilon \) is arbitrary, then \( \theta_R(t, y) \to 0 \) uniformly on \( \{(t, y) \mid -\infty < t < \infty, y \text{ is fixed} \} \) as \( R \to \infty \). Hence the proof is complete.

**Lemma 4.4.** If \( \psi(x, y) \in D(\Omega) \), then

\[ \frac{1}{\pi} k_{a,b}(t, b) \int_{-\infty}^{\infty} [e^{\sigma(t-x)} \psi(x,y) - \psi(t, \tau)] y^{-a-\beta} T_N(y, \tau) \frac{\sin R(x-t)}{(x-t)} \, dx \, dy \]
converges to zero uniformly as $R, N \to \infty$ for all $(t, \tau) \in (-\infty, \infty) \times (0, 1)$, where the support of $\psi(x, y)$ is contained in $[A, B] \times [a', b']$. where $-\infty < A < B < \infty, 0 < a' < b' < 1$.

**Proof.** Let us divide the interval $(-\infty, \infty) \times (0, 1)$ into four disjoint sets $[(-\infty, A) \cup (B, \infty)] \times (0, 1), (A, B) \times (b', 1), (A, B) \times (0, a')$ and $[A, B] \times [a', b']$. For $(t, \tau) \in [(-\infty, A) \cup (B, \infty)] \times (0, 1)$, $\psi(t, \tau) = 0$. Since $\psi(t, \tau)$ is supported by $[A, B] \times [a', b']$. Therefore

\[
\frac{1}{\pi} k_a(t) \int_{-\infty}^{b} \int_{a}^{\infty} e^{\sigma(x-i)} \psi(x, y) - \psi(t, \tau) y^{-\alpha - \beta} T_N(y, \tau) \frac{\sin R(x-t)}{(x-t)} dxdy
\]

\[
= \frac{1}{\pi} k_a(t) \int_{-\infty}^{b} \int_{a}^{\infty} e^{\sigma(x-i)} \psi(x, y) y^{-\alpha - \beta} T_N(y, \tau) \frac{\sin R(x-t)}{(x-t)} dxdy.
\]

In view of Lemma 4.3 as $R \to \infty$, this integral reduces to

\[
\frac{1}{\pi} k_a(t) \int_{a}^{b} y^{-\alpha - \beta} \psi(t, y) T_N(y, \tau) dy.
\]

Thus to show that for fixed $t$ and $0 < y < 1$,

\[
\lim_{N \to \infty} \frac{1}{\pi} k_a(t) \int_{a}^{b} y^{-\alpha - \beta} \psi(t, y) T_N(y, \tau) dy = 0
\]

(4.14) uniformly for all $(t, z)$. Since $\psi(t, y) \in D(\Omega_\tau), \Omega_\tau = \{(t, y) | t$ is fixed, $0 < y < 1\}$, then $\psi(t, y)$ is bounded say by $K$;

\[
k_a(t) \int_{a}^{b} y^{-\alpha - \beta} T_N(y, \tau) dy \leq K \tau \int_{a}^{b} \psi(t, y) y^{-\alpha - \beta} T_N(y, \tau) dy.
\]

In view of the analogue of Riemann Lebesgue lemma [6, p. 589], for given $\epsilon > 0$ there exists a positive integer $N_0$ such that $\int_{a}^{b} y^{-\alpha - \beta} T_N(y, \tau) dy \leq \frac{8e_\tau^2 \epsilon}{\pi c_\tau^2 (2 - \tau - b') \tau^{-2}}$ for all $N \geq N_0$, which is bounded by $c_\tau (1 - b') C^{-\alpha - \beta}$. Therefore, for all $N \geq N_0$ and for all $(t, \tau) \in [(-\infty, A) \cup (B, \infty)] \times (0, 1)$,

\[
k_a(t) \int_{a}^{b} y^{-\alpha - \beta} T_N(y, \tau) dy \leq \frac{c_\tau \epsilon}{(1 - b')} \leq \frac{c_\tau \epsilon}{(1 - b')}
\]

(4.15) since $b \geq 1/2$. Hence as $\epsilon$ is arbitrary, (4.15) is as stated above. Thus

\[
\frac{1}{\pi} k_a(t) \int_{-\infty}^{b} \int_{a}^{\infty} e^{\sigma(x-i)} \psi(x, y) - \psi(t, \tau) y^{-\alpha - \beta} T_N(y, \tau) \frac{\sin R(x-t)}{(x-t)} dxdy \to 0
\]

(4.16) as $R, N \to \infty$, uniformly for all $(t, \tau) \in [(-\infty, A) \cup (B, \infty)] \times (0, 1)$.

In a similar manner, it can be proved that for $(t, \tau) \in (A, B) \times (b', 1)$ and $(t, \tau) \in (A, B) \times (0, a')$;

\[
\frac{1}{\pi} k_a(t) \int_{a}^{\infty} \int_{-\infty}^{b} e^{\sigma(x-i)} \psi(x, y) - \psi(t, \tau) y^{-\alpha - \beta} T_N(y, \tau) \frac{\sin R(x-t)}{(x-t)} dxdy \to 0
\]

(4.17) uniformly as $R, N \to \infty$.

Next to show that

\[
\frac{1}{\pi} k_a(t) \int_{a}^{b} \int_{-\infty}^{\infty} e^{\sigma(x-i)} \psi(x, y) - \psi(t, \tau) y^{-\alpha - \beta} T_N(y, \tau) \frac{\sin R(x-t)}{(x-t)} dxdy \to 0
\]
uniformly as \( R, N \to \infty \) for all \((t, \tau) \in [A, B] \times [a', b']\). Now

\[
\frac{1}{\pi} k_{a}^{(t)} \tau \int_{-\infty}^{b'} \int_{a'}^{b'} \left[ e^{\alpha (x-y)} \psi(x, y) - \psi(t, \tau) \right] y^{-\alpha - \beta} T_{N}(y, \tau) \frac{\sin R(x-t)}{(x-t)} \, dx \, dy
\]

\[
\to k_{a}^{(t)} \tau \int_{a'}^{b'} \left[ \psi(t, y) - \psi(t, \tau) \right] y^{-\alpha - \beta} T_{N}(y, \tau) \, dy
\]
as \( R \to \infty \).

Hence to show that for fixed \( t \) and \( a' < \tau < b' \); \( k_{a}^{(t)} \tau \int_{a'}^{b'} \left[ \psi(t, y) - \psi(t, \tau) \right] y^{-\alpha - \beta} T_{N}(y, \tau) \, dy \to 0 \)
as \( N \to \infty \) uniformly for all \((t, \tau)\). Let \( F(y, t, \tau)(y^{2} - \tau^{2}) = y^{-\alpha - \beta} \left[ \psi(t, y) - \psi(t, \tau) \right] \) for \( 0 < y < 1, 0 < \tau < 1 \), and \( t \) is fixed. Define function:

\[
G(y, t, \tau) = F(y, t, \tau), \, y = \tau
\]

\[
= \frac{y^{-\alpha - \beta} D \psi(t, y)}{2y^{2}}, \, y \neq \tau, D = \frac{\partial}{\partial y}.
\]

\( G(y, t, \tau) \) is continuous of \( y, t \) and \( \tau \) in the domain \( \{ t \text{ is fixed, } 0 < y < 1, 0 < \tau < 1 \} \).

Now

\[
\int_{a'}^{b'} y^{-\alpha - \beta} \left[ \psi(t, y) - \psi(t, \tau) \right] T_{N}(y, \tau) \, dy
\]

\[
= \int_{a'}^{b'} F(y, t, \tau)(y^{2} - \tau^{2}) T_{N}(y, \tau) \, dy
\]

\[
= \int_{a'}^{b'} G(y, t, \tau)(y^{2} - \tau^{2}) T_{N}(y, \tau) \, dy
\]
as the value of the integral remains unchanged by replacing expression \( F(y, t, \tau)(y^{2} - \tau^{2}) \) by \( G(y, t, \tau)(y^{2} - \tau^{2}) \).

Divide the interval \( a' \leq \tau \leq b' \) in to \( p \) equal parts by the points \( a' = y_{0}, y_{1}, ..., y_{p} = b' \) and after choosing positive number \( \varepsilon \), \( p \) is so large that \( \sum_{m=1}^{p} (U_{m} - L_{m}) (y_{m} - y_{m-1}) < \varepsilon \), where \( U_{m} \) and \( L_{m} \) are upper and lower bounds of \( G(y, t, \tau) \) in \( y_{m-1} \leq y \leq y_{m}, a' \leq \tau \leq b' \). Let \( G(y, t, \tau) = G(y_{m-1}, t, \tau) + w_{m}(y, t, \tau) \) for \( y_{m-1} \leq y \leq y_{m}, a' \leq \tau \leq b' \) so that \( |w_{m}(t, y, \tau)| \leq U_{m} - L_{m} \). Using uniform continuity of the function \( G(y, t, \tau) \) over the region \( \alpha' \leq y \leq b', a' \leq \tau \leq b' \) and following the lines in the proof of the analogue of Riemann Lebesgue lemma [6, p. 589], for an arbitrary \( \varepsilon > 0 \) we get a positive integer \( N_{1} \) such that

\[
\int_{a'}^{b'} y^{-\alpha - \beta} G(y, t, \tau)(y^{2} - \tau^{2}) T_{N}(y, \tau) \, dy \leq \frac{c' \varepsilon}{(1-b')^{2}}
\]

for all \( N \geq N_{1} \). Hence for all \((t, \tau)\), \( t \) is fixed and \( a' \leq \tau \leq b' \)

\[
k_{a}^{(t)} \tau \int_{a'}^{b'} \left[ \psi(t, y) - \psi(t, \tau) \right] y^{-\alpha - \beta} T_{N}(y, \tau) \, dy \leq \frac{c' \varepsilon}{(1-b')^{2}} \leq \frac{c' \varepsilon}{(1-b')^{2}} \leq \frac{c' \varepsilon}{(1-b')^{2}} \leq c \geq \frac{1}{2}
\]

Since \( \varepsilon \) is arbitrary,

\[
\frac{1}{\pi} k_{a}^{(t)} \tau \int_{a'}^{b'} \left[ \psi(t, y) - \psi(t, \tau) \right] y^{-\alpha - \beta} T_{N}(y, \tau) \, dy \to 0 \text{ as } N \to \infty
\]
uniformly for all fixed $t$ and $\tau \in [a', b']$.

Thus
\[
\frac{1}{\pi} k_\alpha^{\tau' t} \int_{-\infty}^{\infty} \int_{a}^{b} \left[ e^{\sigma(x-t)} \psi(x, y) - \psi(t, \tau) \right] y^{-\alpha-\beta} T_N(y, \tau) \frac{\sin R(x-t)}{(x-t)} \, dx \, dy \to 0
\]  \tag{4.21}

uniformly for all $(t, \tau) \in [A, B] \times [a', b']$ as $R, N \to \infty$. Combining (4.15), (4.20) and (4.21), the lemma yields.

**Lemma 4.5.** For $\phi(x, y) \in D(\Omega)$, $0 < a' < b' < 1$

\[
\varphi(s, \lambda_m) = \int_{a}^{b} e^{-sx} \int_{-\infty}^{\infty} y^{-\alpha-\beta} \mathcal{J}_{a,\beta}(\lambda_m y) \phi(x, y) \, dx \, dy
\]  \tag{4.22}

\[
\tau^{-\alpha-\beta} \mathcal{M}_{R, N}(t, \tau) = \tau^{-\alpha-\beta} \left[ \sum_{m=1}^{N} \frac{\mathcal{J}_{a,\beta}(\lambda_m y)}{\lambda_m} \int_{-R}^{R} e^{-\sigma x} \mathcal{J}_{a,\beta}(\lambda_m \tau) \phi(s, \lambda_m) \, dw \right]
\]  \tag{4.23}

converges in $LH_{a,\beta,\alpha,\beta}$ to $\tau^{-\alpha-\beta} \phi(t, \tau)$ as $R, N \to \infty$ for all $(t, \tau) \in (-\infty, \infty) \times (0, 1)$

**Proof.** Since the integrand in (4.22) is a smooth function and $\phi$ is of bounded support, differentiate under the integral sign, and obtain

\[
D_k^\alpha \Delta_{a,\beta,\tau}^{k'} \left[ \tau^{\alpha-\beta} \tau^{-\alpha-\beta} \mathcal{M}_{R, N}(t, \tau) \right] = \sum_{m=1}^{N} \frac{1}{\lambda_m} \mathcal{J}_{a,\beta,\tau}(\lambda_m) \left[ \frac{1}{2\pi} \int_{-R}^{R} e^{-\sigma x} \mathcal{J}_{a,\beta}(\lambda_m \tau) \phi(s, \lambda_m) \, dw \right]
\]

\[
= \sum_{m=1}^{N} \frac{1}{\lambda_m} \mathcal{J}_{a,\beta,\tau}(\lambda_m) \left[ \frac{1}{2\pi} \int_{-R}^{R} e^{-\sigma x} \mathcal{J}_{a,\beta}(\lambda_m \tau) \phi(s, \lambda_m) \, dw \right]
\]

Now $(-1)^{k'} \left\{ \int_{-\infty}^{\infty} D_k^\alpha \Delta_{a,\beta,\tau}^{k'} e^{-\sigma x} \mathcal{J}_{a,\beta}(\lambda_m \tau) \phi(s, \lambda_m) \, dx \right\}$ up on integrating by parts, the inner integral $k'$ times and since $\phi(x, y)$ is of compact support this integral reduces to

\[
\int_{a}^{b} y^{-\alpha-\beta} \Delta_{a,\beta,\tau}^{k'} \mathcal{J}_{a,\beta}(\lambda_m y) \left[ \int_{-\infty}^{\infty} D_k^\alpha e^{-\sigma x} \phi(x, y) \, dx \right] \, dy.
\]

Again integrating by parts $2k'$ times follows

\[
\int_{-\infty}^{\infty} \int_{a}^{b} \phi(x, y) D_k^\alpha \Delta_{a,\beta,\tau}^{k'} y^{-\alpha-\beta} e^{-\sigma x} \mathcal{J}_{a,\beta}(\lambda_m y) \, dx \, dy.
\]

Therefore

\[
D_k^\alpha \Delta_{a,\beta,\tau}^{k'} \left[ \tau^{\alpha-\beta} \tau^{-\alpha-\beta} \mathcal{M}_{R, N}(t, \tau) \right] = \frac{1}{2\pi} \sum_{m=1}^{N} \frac{1}{\lambda_m} \mathcal{J}_{a,\beta,\tau}(\lambda_m) \left[ \int_{-R}^{R} e^{-\sigma x} \mathcal{J}_{a,\beta}(\lambda_m \tau) \times \left\{ \int_{-\infty}^{\infty} \int_{a}^{b} \phi(x, y) D_k^\alpha \Delta_{a,\beta,\tau}^{k'} y^{-\alpha-\beta} e^{-\sigma x} \mathcal{J}_{a,\beta}(\lambda_m y) \, dx \, dy \right\} \right] \, dw.
\]
Changing the order of integration, it is
\[
D^k \Delta^k_{\alpha,-\beta} \left[ \tau^{a+\beta} \tau^{-a-\beta} M_{R,N} (t, \tau) \right] =
\]
\[
= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \left[ \int_{a'}^{b'} \int_{-\infty}^{\infty} D^k \Delta^k_{\alpha,-\beta} \phi(x, y) y^{-a-\beta} T_N (y, \tau) \left( \int_{-\infty}^{\infty} e^{-\sum_{m=1}^{N} g_{\alpha,\beta} (\lambda_m \tau) \text{d}w} \right) dx \right] dy \right\}
\]
\[
= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \left[ \int_{a'}^{b'} \int_{-\infty}^{\infty} D^k \Delta^k_{\alpha,-\beta} \phi(x, y) y^{-a-\beta} T_N (y, \tau) \frac{\sin R(x-t)}{(x-t)} e^{-\sigma(x-t)} dx \right] dy \right\}.
\]
Hence from Lemma 4.2 as \( R, N \to \infty \)
\[
D^k \Delta^k_{\alpha,-\beta} \tau^{a+\beta} \left[ \tau^{-a-\beta} M_{R,N} (t, \tau) - \tau^{-a-\beta} \phi(t, \tau) \right] =
\]
\[
= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \left[ \int_{a'}^{b'} \int_{-\infty}^{\infty} e^{-\sum_{m=1}^{N} g_{\alpha,\beta} (\lambda_m \tau) \text{d}w} \left( D^k \Delta^k_{\alpha,-\beta} \phi(x, y) - D^k \Delta^k_{\alpha,-\beta} \phi(t, \tau) \right) \right] y^{-a-\beta} T_N (y, \tau) \frac{\sin R(x-t)}{(x-t)} dx \right\}
\]
\[
= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \left[ \int_{a'}^{b'} \int_{-\infty}^{\infty} e^{-\sum_{m=1}^{N} g_{\alpha,\beta} (\lambda_m \tau) \text{d}w} \left( \psi(x, y) - \psi(t, \tau) \right) \right] y^{-a-\beta} T_N (y, \tau) \frac{\sin R(x-t)}{(x-t)} dx \right\},
\]
where \( \psi(x, y) = D^k \Delta^k_{\alpha,-\beta} \left[ \phi(x, y) \right] \) which is again a member of \( D(\Omega) \) with support contained in \( [A, B] \times [a', b'] \). Hence it suffices to show that
\[
\frac{1}{\pi} k^{(m)} \tau \int_{-\infty}^{\infty} \int_{a'}^{b'} e^{-\sum_{m=1}^{N} g_{\alpha,\beta} (\lambda_m \tau) \text{d}w} \left( \psi(x, y) - \psi(t, \tau) \right) y^{-a-\beta} T_N (y, \tau) \frac{\sin R(x-t)}{(x-t)} dx \right\}
\]
converges to zero as \( R, N \to \infty \) uniformly for all \( (t, \tau) \in (-\infty, \infty) \times (0, 1) \). This is true in view of Lemma 4.4, and thus the proof of Lemma 4.5 is complete.

5. THE MAIN THEOREM.

Let \( f(x, y) \) be a generalized FGLHCT transformable function and \( F(s, \lambda_m) \) the generalized FGLHCT of \( f \) as defined by \( F(s, \lambda_m) = \left\langle f(x, y), e^{-\sum_{m=1}^{N} g_{\alpha,\beta} (\lambda_m \tau) \text{d}w} \right\rangle \) for \( s \in \Delta_f \) and \( \{\lambda_m\} \), the positive zeros of \( g_{\alpha,\beta}(z) \).

Then in the sense of convergence in \( D'(\Omega) \)
\[
f(x, y) = \lim_{R,N \to \infty} \frac{1}{2\pi i} \sum_{m=1}^{N} \frac{g_{\alpha,\beta}(\lambda_m \tau)}{\lambda_m} \frac{\sum_{m=1}^{N} \int_{-\infty}^{\infty} e^{\sigma(x-t)} F(s, \lambda_m) ds \phi(x, y)}{\sum_{m=1}^{N} \int_{-\infty}^{\infty} e^{\sigma(x-t)} F(s, \lambda_m) ds \phi(x, y)}
\]
\[
= \left\langle f(t, \tau), \phi(t, \tau) \right\rangle
\]
where \( \sigma \) is any fixed number \( \sigma_f < \sigma < \rho_f \).

**Proof.** Let \( \phi(x, y) \) be an arbitrary member of \( D(\Omega) \). To show that
\[
\left\langle \frac{1}{2\pi i} \sum_{m=1}^{N} \frac{g_{\alpha,\beta}(\lambda_m \tau)}{\lambda_m} \frac{\sum_{m=1}^{N} \int_{-\infty}^{\infty} e^{\sigma(x-t)} F(s, \lambda_m) ds \phi(x, y)}{\sum_{m=1}^{N} \int_{-\infty}^{\infty} e^{\sigma(x-t)} F(s, \lambda_m) ds \phi(x, y)} \right\rangle = \left\langle f(t, \tau), \phi(t, \tau) \right\rangle
\]
as \( R, N \to \infty \).

Since \( \phi(x, y) \in D(\Omega) \) iff \( y\phi(x, y) \in D(\Omega) \), then (5.2) will be equivalent to showing that
\[
\left\langle \frac{1}{2\pi i} \sum_{m=1}^{N} \frac{g_{\alpha,\beta}(\lambda_m \tau)}{\lambda_m} \frac{\sum_{m=1}^{N} \int_{-\infty}^{\infty} e^{\sigma(x-t)} F(s, \lambda_m) ds \phi(x, y)}{\sum_{m=1}^{N} \int_{-\infty}^{\infty} e^{\sigma(x-t)} F(s, \lambda_m) ds \phi(x, y)} \right\rangle = \left\langle f(t, \tau), \phi(t, \tau) \right\rangle
\]
as \( R, N \to \infty \).

As \( \phi(x, y) \in D(\Omega) \), assume that the support of \( \phi(x, y) \) is contained in \( [A, B] \times [a', b'] \) where \(-\infty < A < B < \infty, 0 < a' < b' < 1\).
As \( \frac{1}{2\pi i} \sum_{m=1}^{N} \frac{\mathcal{J}_{\alpha,\beta}(\lambda_m y)}{\lambda_m} \int_{-R}^{R} e^{s\tau} F(s, \lambda_m) ds \) is locally integrable and since \( y^{-\alpha - \beta} \phi(x, y) \in D(\Omega) \) then without limit notation (4.3) can be written as
\[
\int_{-\infty}^{\infty} \int_{a}^{b} y^{-\alpha - \beta} \phi(x, y) \left[ \frac{1}{2\pi i} \sum_{m=1}^{N} \frac{\mathcal{J}_{\alpha,\beta}(\lambda_m y)}{\lambda_m} \int_{-R}^{R} e^{s\tau} F(s, \lambda_m) ds \right] dx dy.
\]
Substituting \( s = \sigma + i\nu \), leads to
\[
\int_{-\infty}^{\infty} \int_{a}^{b} y^{-\alpha - \beta} \phi(x, y) \left[ \frac{1}{2\pi} \sum_{m=1}^{N} \frac{\mathcal{J}_{\alpha,\beta}(\lambda_m y)}{\lambda_m} \int_{-R}^{R} e^{s\tau} F(s, \lambda_m) dw \right] dx dy.
\]
Since \( \phi(x, y) \) has a compact support and the integrand is a continuous function of \( (x, y, \nu) \), can interchange the order of integration as in [7].
\[
\frac{1}{2\pi} \sum_{m=1}^{N} \frac{\mathcal{J}_{\alpha,\beta}(\lambda_m y)}{\lambda_m} \left[ \int_{-R}^{R} \left( f(t, \tau), e^{-s\tau} \mathcal{J}_{\alpha,\beta}(\lambda_m \tau) \right) \phi(s, \lambda_m) ds \right] dx dy.
\]
Now by Lemma 5.1,
\[
\frac{1}{2\pi} \sum_{m=1}^{N} \frac{\mathcal{J}_{\alpha,\beta}(\lambda_m y)}{\lambda_m} \left[ \int_{-R}^{R} \left( f(t, \tau), e^{-s\tau} \mathcal{J}_{\alpha,\beta}(\lambda_m \tau) \right) \phi(s, \lambda_m) ds \right] dx dy.
\]
Since \( f \) is a continuous linear functional,
\[
\frac{1}{2\pi} \sum_{m=1}^{N} \frac{\mathcal{J}_{\alpha,\beta}(\lambda_m y)}{\lambda_m} \left[ f(t, \tau), \frac{1}{2\pi} \int_{-R}^{R} e^{-s\tau} \mathcal{J}_{\alpha,\beta}(\lambda_m \tau) \phi(s, \lambda_m) ds \right] dx dy.
\]
Because \( f \in LH_{a,\alpha,\beta}' \) and in view of Lemma 4.5, the last expression tends to \( \left\langle f(t, \tau), e^{-s\tau} \phi(t, \tau) \right\rangle \) as \( R, N \to \infty \). This completes our proof of the main theorem.

6. UNIQUENESS THEOREM6.1. If \( Lh_{\alpha,\beta}(f) = F(s, \lambda_m) \) and \( Lh_{\alpha,\beta}(g) = G(s, \lambda_m) \) for all \( s \in \Omega_f = \{ s / \sigma_f < \text{Re}(s) < \rho_f \} \) and \( s \in \Omega_g = \{ s / \sigma_g < \text{Re}(s) < \rho_g \} \) and \( \lambda_m \) be the positive zeros of \( \mathcal{J}_{\alpha,\beta}(x) \), if \( \Omega_f \cap \Omega_g \neq 0 \), and if \( F(s, \lambda_m) = G(s, \lambda_m) \) for \( s \in \Omega_f \cap \Omega_g \), then \( f = g \) and in the sense of equality in \( D' \Omega \).

**Proof.** By above theorem, in the sense of convergence in \( D' \Omega \)
\[ f = \lim_{r,N \to \infty} \left[ \frac{1}{2\pi i} \sum_{m=1}^{N} \zeta_{a,\beta} \left( \lambda_m, y \right) e^{s \lambda_m} \int e^{sx} F(s, \lambda_m) \, ds \right] \]

\[ = \lim_{r,N \to \infty} \left[ \frac{1}{2\pi i} \sum_{m=1}^{N} \zeta_{a,\beta} \left( \lambda_m, y \right) e^{s \lambda_m} \int e^{sx} G(s, \lambda_m) \, ds \right] \]

\[ = g(x, y). \]

Hence \( f = g \) and in the sense of equality in \( D'(\Omega) \).

### 7. Illustration of the inversion formula by means of an example:

Consider the Dirac delta functional \( \delta(t-k, r-v) \) concentrated at a point \((k, v)\), \(-\infty < k < \infty, 0 < v < 1\), since \( \delta(t-k, r-v) \in E'(I) \) which is a subspace of \( LH^r_{a,b,a,\beta}(I) \), then \( \delta(t-k, r-v) \) is a member of \( LH^r_{a,b,a,\beta}(I) \). The FGLHC transform of \( \delta(t-k, r-v) \) is given as \( \mathcal{L}^{\alpha, \beta} \mathcal{R} \mathcal{H}^{\alpha, \beta} \left( \delta(t-k, r-v) \right) = \left\{ \delta(t-k, r-v), e^{-sx} r^{-\alpha, \beta} \zeta_{a,\beta} \left( \lambda_m r \right) \right\} \]

Now for any \( y^{-\alpha, \beta} \phi(x, y) \in D(I) \),

\[ \left\{ \frac{1}{2\pi i} \sum_{m=1}^{N} \zeta_{a,\beta} \left( \lambda_m, y \right) e^{s \lambda_m} \int e^{sx} e^{-sk} v^{-\alpha, \beta} \zeta_{a,\beta} \left( \lambda_m, v \right) ds, y^{-\alpha, \beta} \phi(x, y) \right\} \]

\[ = \lim_{-\infty \to \infty} \int_{-\infty}^{\infty} \sum_{m=1}^{N} \zeta_{a,\beta} \left( \lambda_m, y \right) \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{sx} e^{-sk} v^{-\alpha, \beta} \zeta_{a,\beta} \left( \lambda_m, v \right) ds \right] y^{-\alpha, \beta} \phi(x, y) \, dx \, dy \]

On changing order of integration,

\[ = v^{-\alpha, \beta} \sum_{m=1}^{N} \zeta_{a,\beta} \left( \lambda_m, y \right) \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{sx} \phi(x, y) \, dx \, dy \right] \int_{-\infty}^{\infty} e^{s \lambda_m} \zeta_{a,\beta} \left( \lambda_m, y \right) \phi(x, y) \, dx \, dy \, dw \]

\[ = v^{-\alpha, \beta} \phi(k, v) \] as \( R, N \to \infty \).

For all \((k, v) \in (-\infty, \infty) \times (0, 1)\), but \( v^{-\alpha, \beta} \phi(k, v) = \{ \delta(x-k, y-v), y^{-\alpha, \beta} \phi(x, y) \} \), and therefore the inversion theorem is illustrated.

### 8. Operational calculus:

FGLHC generates an operational calculus by means of which certain differential equation involving generalized functions can be solved. Now differential equations that can be solved by using the FGLHC are of the form:

\[ P \left[ \left( D x \Delta_{a,\beta,\gamma} \right)^{s} \right] u(x, y) = g(x, y) \]  \hspace{1cm} (8.1)

in which \( g(x, y) \) is given member of \( LH^r_{a,b,a,\beta}(I) \) and \( u(x, y) \) is unknown generalized function required to be in \( LH^r_{a,b,a,\beta}(I) \). \( P \) is a polynomial such that \( P \left( s(-\lambda_m) \right) \neq 0; m = 1, 2, 3, \ldots \)

By applying \( \mathcal{L}^{\alpha, \beta} \) to (8.1),
So that
\[
U(s, \lambda_m) = \frac{G(s, \lambda_m)}{P(s(-\lambda_m)^2)}
\]  
where \(U\) and \(G\) are FGLHCT of \(u\) and \(g\) respectively.

Applying the inversion theorem (4.1) to (8.3), \(u(x, y) = \lim_{\kappa,\eta \to \infty} \frac{1}{2\pi i} \sum_{m=1}^{\infty} \frac{P J_{\alpha, \beta}^{\pm} ((\lambda_m y))}{\lambda_m} \int_{\sigma-i \kappa}^{\sigma+i \eta} e^{sx} G(s, \lambda_m) ds\)

with equality in the sense of \(D'(I)\), which is a solution to (8.1). This solution is unique in view of the uniqueness theorem. One can verify that \(u\) as determined in (8.1) is also a solution of differential equation
\[
P \left[ (D_x A_{\alpha, \beta, y})^+ \right] u(x, y) = g(x, y).
\]

9. MAIN PROBLEM:
Let the domain in the cylindrical co-ordinate system be defined by \(-\infty < z < \infty, 0 < r < 1, 0 < t < \infty\). Then assuming axial symmetry shall obtain the conventional function \(v(z, r, t)\) on the domain \(R = \{(z, r, t)\mid -\infty < z < \infty; 0 < r < 1; 0 < t < \infty\}\) that satisfies the heat equation in the cylindrical co-ordinates:
\[
r D^2_r (v) + (1-\alpha-\beta)D_r (v) + r^{-1} \alpha \beta v + D_z^2 (v) = \frac{1}{k} D_t (v)
\]
where real positive constant \(k\) represents the diffusivity of the cylinder’s material, with the following conditions:

i) As \(t \to 0^+\), \(v(z, r, t)\) converges in the sense of \(D'(I)\) to some FGLHCT generalized function \(f(z, r)\) in \(LH_{\alpha, \beta, \alpha, \beta}(I)\).

ii) As \(t \to 1^-\), \(v(z, r, t)\) converges uniformly to zero on \(-\infty < z < \infty, 0 < t < \infty\).

Solution of problem:
Using FGLHCT \(Lh_0\) of order zero to solve this problem. Now equation (9.1) can be written as
\[
k \left[ \Delta_{0,r} (v) + D_z^2 (v) \right] = D_t (v)
\]
where \(\Delta_{0,r} = r D^2_r (v) + (1-\alpha-\beta)D_r (v) + r^{-1} \alpha \beta \).

By applying the zero order FGLHCT \(Lh_0\) to (9.2) and formally interchanging \(Lh_0\) with \(D_t\) and using the condition (ii), converting (9.2) into
\[
k \left[ (-\lambda_m)^2 + s^2 \right] v(s, \lambda_m, t) = D_t v(s, \lambda_m, t)
\]
where \(v(s, \lambda_m, t) = Lh_0 v(z, r, t)\)

Then \(v(s, \lambda_m, t) = A(s, \lambda_m) e^{k(s^2-\lambda_m^2)t}\), where \(A(s, \lambda_m)\) does not depend on \(t\). In view of the initial condition (i), \(A(s, \lambda_m) = F(s, \lambda_m) = \left\{ f(z), r^{-\alpha} e^{-\alpha r^2} g_0(\lambda_m r) \right\}\), and \(\{\lambda_m\}\) are the positive zeros of \(g_{\alpha, \beta}(z)\).

Thus
\[
v(s, \lambda_m, t) = F(s, \lambda_m) e^{k(s^2-\lambda_m^2)t}.
\]
Furthermore, for each \( t > 0 \), the theorem 4.1 states that the function \( F(s, \lambda_m) e^{k(r-i\omega)^2} \) is analytic on \( \Lambda_j = \{ s \mid \sigma_j < \text{Re}(s) < \rho_j \} \) for fixed \( \lambda_m \). In view of theorem 4.1, apply the conventional inverse FGLHCT to get formal solution
\[
v(z, r, t) = \frac{1}{2\pi i} \sum_{m=1}^{\infty} \int_{\lambda_m}^{\lambda_m} \sigma^{e^{i\pi}} F(s, \lambda_m) e^{k(r-i\omega)^2} ds
\]
Verifying that (9.4) is truly a solution of (9.1). First of all it is a fact that 
\[
\lambda_m^{1/2} \approx \left( m + \frac{\alpha + \beta}{2} - \frac{1}{4} \right) \pi^{1/2} \text{ as } m \to \infty, \quad \lambda_m^{1/2} \approx \sqrt{2l(\pi \lambda_m^{1/2})} \text{ as } m \to \infty.
\]
Also for each \( T > 0 \), \( e^{k(r-i\omega)^2} \) uniformly on \( T < t < \infty \). By the theorem (4.2), \( F(s, \lambda_m) \) is bounded \( \lambda_m^{1/2} \) uniformly bounded for \( 0 < r < 1, m = 1, 2, 3, \ldots \). These facts imply that the series (9.4) and the series obtained by applying the operator \( \Delta_{0,r} \), \( D_z^2 \) and \( D_t \) separately under the summation sign of (9.4) converge uniformly on \( R = \{ (z, r, t) \mid -\infty < z < \infty; 0 < r < 1; 0 < t < \infty \} \). Thus applying \( k \left[ \Delta_{0,r}(v) + D_z^2(v) \right] = D_t(v) \) term by term to (9.4) and using the fact that \( \Delta_{0,r} \left[ \lambda_m^{1/2} \right] = -\lambda_m^{1/2} \lambda_m^{1/2} \).

Eqn. (9.4) satisfies differential equation (9.1). The uniform convergence of the series in (9.4) allows to interchange differentiation in (9.1) with summation in (9.4). The uniform convergence of the series in (9.4) leads to the verification of boundary conditions (ii) by taking the limit as \( r \to 1 \) under the summation sign.

Finally it remains to verify the initial conditions (i). It can be verified as follows.

The series (9.4) converges uniformly for all \( z, r, t \). Hence \( v(z, r, t) \) is a continuous function of \( z, r, t \). Therefore it generates a regular generalized function in \( D'(1) \) with \( z, r \) as parameters and for any \( \phi \in D(1) \),
\[
\lim_{t \to 0^+} \left\langle v(z, r, t), \phi(z, r) \right\rangle = \lim_{t \to 0^+} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi i} \sum_{m=1}^{\infty} \int_{\lambda_m}^{\lambda_m} \sigma^{e^{i\pi}} F(s, \lambda_m) e^{k(r-i\omega)^2} ds \phi(z, r) dzdr.
\]
By uniform convergence of series in (9.4) on \( R = \{ (z, r, t) \mid -\infty < z < \infty; 0 < r < 1; 0 < t < \infty \} \) and by using inversion theorem (4.1) \( \lim_{t \to 0^+} \left\langle v(z, r, t), \phi(z, r) \right\rangle = \left\langle f(z, r), \phi(z, r) \right\rangle \). Thus shown that, in the sense of convergence in \( D'(1) \), \( v(z, r, t) \to f(z, r) \) as \( t \to 0^+ \). This completes verification of (9.4) as a solution. This solution is unique in the sense of equality.

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