Computer-Assisted Solution Of The Dbf Equation Using The Hcm For Flow Through A Rectangular Porous Channel Bounded By Differentially Heated Horizontal Plate’s Demonstration Using Mayavi

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Abstract: The system of non-linear algebraic equations arising from the application of the central difference approximation to the fully developed Darcy-Brinkman-Forchheimer flow equation is solved using the continuation method based on the Runge-Kutta45 order method. The fully-developed non-linear flow through a rectangular channel is considered and the influence of the Brinkman, Reynolds and Darcy numbers, the step size and variable viscosity parameter on the numerical results is investigated.

Keywords: Brinkman, Reynolds, Darcy, Forchheimer, Runge-Kutta45, Mayavi

1. INTRODUCTION

The utility of porous media in practical applications is well known at the present time to merit more than a not-so-detailed exposition (see Nield and Bejan[1], Vafai[2], Rudraiah et al[3]). In the light of the above observation we merely reiterate the advocacy of umpteen number of eminent researchers to have boundary and inertia effects in flow equations of porous media. To cite a few recent works, we draw attention to the work of Skjetne and Auriault[4] that provides new insights on steady, non-linear flow in porous media. Also the work of Calmidi and Mahajan[5] presents the non-linear, non-Darcy equation as an excellent candidate for description of flow in metal foam porous media. Khaled and Vafai[6] draw us from extra-corporeal application situations into corporeal flows. Their work suggests a non-linear flow model for high perfused skeletal tissues. In all these applications, and many more, high flow rate and/or high permeability in porous media warrants the quantification of the dependence of flow in the medium due to constant pressure gradient. In literature, the nomenclature attached with this friction is either after the name of Forchheimer [9] or Ergun [10]. We follow the classic mathematical works and prefer use of the name of Forchheimer to Ergun on reasons of maintaining continuity in nomenclature and not adding to the confusion in literature over different model names. In so far as the Darcy friction and viscous shear is concerned, there is no debate at the present time on having two viscosities in the equation- actual fluid viscosity and effective viscosity (see Lauriat and Prasad[11] and Givler and Altobelli [12]). In the present problem we consider the two non-Darcy effects due to inertia and boundary. Poulikakos and Renken [13] performed a numerical study of boundary and inertia effects on porous medium flow and heat transfer. Parang and Keyhani [14] analysed mixed convection in an annular region considering boundary and inertia effects. Hooman [15,16] quite recently has published numerical works pertaining to this non-linear flow model in porous media. In the paper we report solution of the Darcy-Brinkman-Forchheimer equation for fully developed two-dimensional flows through a channel with variable viscosity parameter using the computer assisted homotopy continuation method.

2. MATHEMATICAL FORMULATION FOR A RECTANGULAR CHANNEL OCCUPIED BY A TEMPERATURE SENSITIVE NEWTONIAN LIQUID

The physical system consists of a highly percolative medium of an infinite horizontal extent bounded by differentially heated horizontal plates. It is assumed that the non – Darcy fully-developed flow in the medium due to constant pressure gradient can be described by the Darcy - Brinkman – Forchheimer(DBF) model and so we have
where $\rho$ is the density, $u(y)$ is the axial filter velocity, $p(x)$ is the axial pressure, $\mu$ is the dynamic viscosity, $\bar{\mu}(T)$ is the effective viscosity, $K$ is the permeability and $C_e$ is the dimensionless quadratic drag coefficient. We have chosen to use two viscosities in the above equation as this is in keeping with the current modeling trend [11–12].

The boundary conditions for solving Eq. (1) are

$$u = 0 \quad \text{at} \quad y = 0, h,$$

(2)

which is the ‘no-slip’ condition.

We now non-dimensionalize Eq. (1) using the following definitions.

$$X = \frac{x}{h}, \quad Y = \frac{y}{h}, \quad P = \frac{p}{\rho u_o/h}, \quad U = \frac{u}{-u_o \frac{dP}{dX}},$$

(3)

where $u_o$ is a characteristic velocity. Substituting Eq. (3) into Eqs. (1) – (2), we get

$$\frac{d^2U}{dY^2} - \Lambda Da^2U - F U^2 = -\Lambda$$

(4)

$$U(Y) = 0 \quad \text{at} \quad Y = 0, 1,$$

(5)

where

$$\Lambda = \frac{\mu}{\bar{\mu}} \quad \text{(Brinkman number),}$$

$$Re = -\frac{\rho u_o h}{\mu} \frac{dP}{dX} \quad \text{(Reynolds number)}$$

and
Da = \frac{h}{\sqrt{K}} \quad \text{(Darcy number).}

F = C_f \Lambda \text{Re} \, Da \quad \text{(Forchheimer number)}

At this juncture we note that the Brinkman and Darcy numbers used here are actually the inverse of the classical definitions. We prefer to continue with the above notation in view of the apparent convenience using the same. The definition of the Brinkman, Darcy and Forchheimer numbers in equation (4) is the same as that of Nield et al.[8].

3. FINITE DIFFERENCE APPROXIMATION AND HOMOTOPY CONTINUATION METHOD OF SOLUTION

In the method of solution adopted in the paper we need to discretize the interval of interest \([\varepsilon, 1]\). We do this by using the discrete points

\[ Y_n = \varepsilon + n\Delta Y = \varepsilon + \frac{n(1 - \varepsilon)}{N}, \quad n = 0(1)N. \]  

(14)

In what follows \(U(Y_n)\) is denoted by \(U_n\). We first apply the central difference approximation to the first and order second derivatives in equation (10) and this procedure yields:

\[
\frac{U_{n+1} - 2U_n + U_{n-1}}{(\Delta Y)^2} + \frac{\delta}{\varepsilon + n\Delta Y} \left( \frac{U_{n+1} - U_{n-1}}{2\Delta Y} \right) - \Lambda Da^2 U_n - F U_n^2 + \Lambda = 0, \quad n = 0(1)N - 1. 
\]  

(15)

Rearranging equation (15) we get a system of \(N\) non-linear algebraic equations in the form:

\[
f_n(U) = F U_n^2 - N^2 \left(1 + \frac{\delta}{2n}\right) U_n + \left(2N^2 + A\right) U_n - N^2 \left(1 - \frac{\delta}{2n}\right) U_{n-1} - \Lambda = 0, \quad n = 0(1)N - 1, 
\]  

(16)

with

\[
\begin{aligned}
U_0 &= 0 \quad \text{(no-slip condition),} \\
U_N &= 0 \quad \text{(no-slip condition).}
\end{aligned} 
\]  

(17)

In equation (16), \(A\) is given by

\[ A = \Lambda Da^2. \]  

(18)

Let \(\vec{U}\) be the unknown solution of the system (17). To obtain this solution we shall later consider a family of problems to be described using a homotopy parameter \(p \in [0,1]\). Henceforth, to bring in ‘\(p\)’, we use a suggestive notation \(\vec{U}(Y; p)\) in place of \(\vec{U}(Y)\) and \(f_n(U; p)\) in place of \(f_n(U)\). With the intention of obtaining \(\vec{U}(Y; 1) = \vec{U}^*\) from \(\vec{U}(Y; 0)\), an assumed initial approximation to the system (16), we define the following mapping

\[ G: [0,1] \times R^N \rightarrow R^N, \]  

(19)

by

\[
G(U; p) = pF(U) + (1 - p)[F(U) - F(U(0))] 
\]  

(20)

where

\[ \vec{G} = [f_0, f_1, \ldots, f_{N-1}] \]  

and
\[ U = [U_0, U_1, \ldots, U_{N-1}]^T. \]

In \( U \), the \( U_n \)s are actually \( U_n(Y; p) \)s, unless otherwise mentioned. Let us assume \( U(Y; p) \) is the unique solution of

\[ G(U; p) = 0, \tag{21} \]

for each \( p \in [0,1] \).

Differentiating equation (21) with respect to \( p \), we get

\[ \frac{\partial G}{\partial p} + \frac{\partial G}{\partial U} \frac{\partial U}{\partial p} = 0, \tag{22} \]

where

\[
\frac{\partial G}{\partial U}(U; p) = \begin{bmatrix}
\frac{\partial f_0}{\partial U_0} & \frac{\partial f_0}{\partial U_1} & \cdots & \frac{\partial f_0}{\partial U_{N-1}} \\
\frac{\partial f_1}{\partial U_0} & \frac{\partial f_1}{\partial U_1} & \cdots & \frac{\partial f_1}{\partial U_{N-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_{N-1}}{\partial U_0} & \frac{\partial f_{N-1}}{\partial U_1} & \cdots & \frac{\partial f_{N-1}}{\partial U_{N-1}}
\end{bmatrix}_j(U_0, U_1, \ldots, U_{N-1})
\]

and using equation (20), we have

\[ \frac{\partial G}{\partial p}(U; p) = F(U(Y, 0)) = [f_0, f_1, \ldots, f_{N-1}]^T_{j(U_0, U_1, \ldots, U_{N-1})}. \]

In the above \( U_0[0] \) is the assumed initial approximation of system (17). Rearranging equation (22), we get a system of \( N \) differential equations:

\[ \frac{\partial U}{\partial p} = -\left[ \frac{\partial G}{\partial U}(U; p) \right]^{-1} \frac{\partial G}{\partial p}(U; p). \tag{23} \]

To solve the system (23), subject to condition (12), by the explicit, classical Runge-Kutta method of four slopes we partition the range of \( p \), namely \([0,1]\), into \( M \) sub-intervals with the mesh points

\[ p_j = j \Delta p = \frac{j}{M}, \quad j = 0(1)M. \tag{24} \]

The results obtained by solving equations (23), subject to conditions (12), by the RKF45- method are documented in figures (2)- (4).

4. RESULTS AND DISCUSSION

As made known quite explicitly in the introduction it is the intention of the paper to propose the computer-assisted continuation method for solving a non-linear, non-Darcy equation with quadratic drag. Before we embark on a discussion of the solution we note here that the definition of Brinkman and Darcy numbers as used in the paper is inverse of the classical definitions. We now move on to discuss the results obtained in the paper.

The results of the present paper indicate that the effect of the Forchheimer number, \( F \) on the flow velocity becomes weak for low-percolation media. Figure 2 illustrates the fact that the velocity is constant near to lower plates and increase the velocity near to the upper plates with the fixed value of Darcy number for rectangular channel flow through porous media. Figure 3 illustrates the fact that the velocity is constant near to lower plates and increase the velocity near to the upper plates with the fixed value of Brinkman number for rectangular channel flow through porous media.
Figure 4 illustrates the fact that the velocity is constant near to lower plates and increase the velocity near to the upper plates with the fixed value of Forchheimer number for rectangular channel flow through porous media. The excellent results on boundary and inertia effects on flow velocity speak about the utility of the method in capturing detailed flow features. It is important to mention here that the method succeeds in giving the required solution for some parameters’ combination when shooting technique, based on Runge-Kutta-Fehlberg45 and modified Newton-Raphson methods, fails for large values of $F$.

As a general observation we note that:

1. $K_{\text{density packed}} < K_{\text{sparse packed}}$

For slow speed flows:

Darcy model

$$-\frac{\mu(T)}{k} u = \frac{dp}{dx}$$

Brinkman model

$$\frac{d}{dy} \left( \frac{\mu(T)}{k} \frac{du}{dy} \right) - \frac{\mu(T)}{k} u = \frac{dp}{dx}$$

Where $k \rightarrow \text{Permeability}, \mu \rightarrow \text{Dynamic viscosity}, \mu \rightarrow \text{Effective viscosity}$

$C_F \rightarrow \text{Dimensionless quadratic co-efficient.}$

For high speed flows:

Darcy-Forchheimer model

$$-\frac{\mu(T)}{k} u - \frac{\rho C_F}{\sqrt{k}} u^2 = \frac{dp}{dx}$$

Brinkman-Forchheimer model

$$\frac{d}{dy} \left( \frac{\mu(T)}{k} \frac{du}{dy} \right) - \frac{\mu(T)}{k} u - \frac{\rho C_F}{\sqrt{k}} u^2 = \frac{dp}{dx}$$

$Da = \frac{h}{\sqrt{k}}$

Where $h$ is half the height of the porous flows.

5. REFERENCES


Table 1: No. of equations required for convergence, for different parameters, efficiency of the algorithm is $O(n^3)$.

<table>
<thead>
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<th>$\Lambda$</th>
<th>$F$</th>
<th>$Da$</th>
<th>No. of equations required for convergence</th>
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<td>1.2</td>
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</table>

Fig 2: Plot of $U(Y)$ vs. $Y$ for constant viscosity $\nu=0$ with fixed values of other parameters $Da$, $F$ and $\Lambda$ for a rectangular channel flow.
Fig 3: Plot of $U(Y)$ vs. $Y$ for variable viscosity $v$ with fixed values of $\Lambda$, $F$ and $Da$ for a rectangular channel flow.

Fig 4: Plot of $U(Y)$ vs. $Y$ for variable viscosity with different values of $\Lambda$ and fixed values of $F$ and $Da$ for channel flow.
Fig 5: Plot of $U(Y)$ vs. $Y$ for variable viscosity with different values of $Da$ and fixed values of $F$, $\Lambda$ and $Da$ for channel flow with visualization through mayavi.

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