A New Composite Implicit Iteration Process for Generalized Asymptotically quasi-nonexpansive Mappings in Banach spaces
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Abstract: The purpose of this paper is to prove the strong convergence of a new modified composite implicit iterative process with errors for two finite families of generalized asymptotically quasi-nonexpansive mappings in framework of Banach space. Our results extendsome known results.


Key Words : Generalized asymptotically quasi-nonexpansive mapping, composite implicit iterative scheme, uniformly continuous, Banach space.

1 Introduction
Throughout this paper, we assume that $E$ is a real normed linear space and $C$ be a nonempty subset of $E$. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$. The important generalization of the class of nonexpansive mappings is the class of asymptotically nonexpansive mappings. The concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [6]. A mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a real sequence $\{h_n\} \subset [0, \infty)$ with $h_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + h_n)\|x - y\|, \ \forall x, y \in C \text{ and } n \geq 1$$

A mapping $T : C \to C$ is called uniformly L-Lipschitzian if there exists a real number $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \ \forall x, y \in C \text{ and } n \geq 1.$$ 

Every nonexpansive mapping is uniformly L-Lipschitzian with $L = 1$. Every asymptotically nonexpansive mapping with sequence $\{k_n\}$ is also uniformly L-Lipschitzian with $L = \sup_{n \in \mathbb{N}} h_n$.

A mapping $T : C \to C$ is called to be asymptotically quasi-nonexpansive mappings if there exists a real sequence $\{h_n\} \subset [0, \infty)$ with $h_n \to 0$, as $n \to \infty$ such that,

$$\|T^n x - p\| \leq (1 + h_n)\|x - p\|, \ \forall x \in C, p \in F(T), \ n \geq 1.$$ 

where $F(T)$ denotes the set of fixed points of $T$. It is clear from this definition that every asymptotically nonexpansive mapping with a fixed point is asymptotically quasi-nonexpansive. But converse
does not hold. The class of asymptotically quasi-nonexpansive mapping was introduced as a generalization of the class of asymptotically nonexpansive mappings by Liu Qihou [12]. Recently, convergence theorems for nonexpansive mapping and asymptotically nonexpansive mapping have been studied extensively by several authors [7, 13, 14]. The modified Ishikawa iteration with errors [8] and the modified Mann iteration with errors [9], they have studied methods for the iterative approximation of fixed points of asymptotically nonexpansive mappings.

In 2001, Xu and Ori [10] introduced the following implicit iteration process for a finite family of nonexpansive mappings \( \{T_1, T_2, \ldots, T_N\} \) with \( \{\alpha_n\} \) a real sequence in \((0, 1)\), and an initial point \( x_0 \in C \):

\[
x_1 = \alpha_1 x_0 + (1 - \alpha_1)T_1 x_1 \\
x_2 = \alpha_2 x_1 + (1 - \alpha_2)T_2 x_2 \\
\vdots \\
x_N = \alpha_N x_{N-1} + (1 - \alpha_N)T_N x_N x_N \\
x_{N+1} = \alpha_{N+1} x_N + (1 - \alpha_{N+1})T_1 x_{N+1}
\]

which can be written in the following compact form:

\[
x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n, \quad n \geq 1,
\]

where \( T_n = T_n(\text{mod}N) \) (here the \( \text{mod}N \) function takes values in \( \{1, 2, \ldots, N\} \)). They also proved the weak convergence of this process to a common fixed point of nonexpansive mappings in a Hilbert space.


The purpose of this work to prove strong convergence theorem for a new composite implicit iteration process with errors for two finite families of generalized asymptotically quasi-nonexpansive mappings in Banach space.

## 2 Preliminaries

In this paper we need the following definitions and lemmas:

**Definition 2.1** [15] Let \( E \) be a real normed linear space and \( C \) be a nonempty subset of \( E \). A mapping \( T : C \to C \) is said to be generalized asymptotically nonexpansive mapping if \( F(T) \neq \emptyset \) and there exist sequences of real numbers \( \{h_n\}, \{\lambda_n\} \) with \( \lim_{n \to \infty} h_n = 0 = \lim_{n \to \infty} \lambda_n \) such that following inequality holds:

\[
\|T^n x - p\| \leq (1 + h_n)\|x - p\| + \lambda_n, \quad \forall x \in C, \quad p \in F(T), \quad n \geq 1.
\]

We give following Example 2.2 below shows:
Example 2.2 Let \( C = [\frac{-1}{2}, \frac{1}{2}] \) and define \( Tx = x \cos(\frac{1}{x}) \), if \( x \neq 0 \) and \( Tx = 0 \) if \( x = 0 \). Then \( T^n x \to 0 \). Clearly \( F(T) = 0 \). For each fixed \( n \geq 1 \), define
\[
f_n(x) = \|T^n x\| - \|x\|
\]
for \( x \in C \). Set
\[
h_n = \frac{1}{n(n + 1)}, \quad \lambda_n = \sup_{x \in C} f_n(x) \vee 0 = \sup_{x \in C} (\|T^n x\| - \|x\|) \vee 0.
\]
for all \( n \in \mathbb{N} \). Then, we have
\[
\lim_{n \to \infty} h_n = 0, \quad \lim_{n \to \infty} \lambda_n = 0
\]
Thus, for all \( n \geq 1 \), the above inequality yields
\[
\|T^n x\|(1 + h_n) \leq \|x\| + \lambda_n.
\]
Therefore \( T \) is a generalized asymptotically quasi-nonexpansive mapping.

Definition 2.3 Let \( C \) be a nonempty closed convex subset of \( E \) and \( T : C \to C \) be a mapping. \( T \) is said to be semi-compact, if for any bounded sequence \( \{x_n\} \) in \( C \) such that \( \|x_n - T x_n\| \to 0 \) as \( n \to \infty \), then there exists a subsequence \( \{x_{n_k}\} \subset \{x_n\} \) such that \( x_{n_k} \to x^* \in C \).

For every \( \epsilon \) with \( 0 < \epsilon < 2 \), we define the modulus \( \delta_C(\epsilon) \) of convexity of \( C \) by
\[
\delta_C(\epsilon) = \inf \left\{ 1 - \frac{\|x - y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}.
\]
A Banach space \( C \) is said to be uniformly convex if \( \delta_C(\epsilon) > 0 \) for each \( \epsilon \in (0, 2) \).

In this paper, motivated and inspired by Xu and Ori [10], Chang et al.[1], Phubtieng et al.[11], Cho et al.[5], we introduced a composite implicit iteration process with errors for two finite families of generalized asymptotically quasi-nonexpansive mappings \( \{T_1, T_2, \ldots, T_N\} \) and \( \{S_1, S_2, \ldots, S_N\} \) with \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1) \) and \( \alpha_n + \beta_n + \gamma_n = 1 = \alpha_n + \beta_n + \gamma_n \). Let \( \{u_n\}, \{v_n\} \) are bounded sequences in \( C \) and an initial point \( x_0 \in C \) as follows:
\[
x_1 = \alpha_1 S_1 x_0 + \beta_1 T_1(\alpha_1 T_1 x_1 + b_1 S_1 x_1 + c_1 v_1) + \gamma_1 u_1
\]
\[
x_2 = \alpha_2 S_2 x_1 + \beta_2 T_2(\alpha_2 T_2 x_2 + b_2 S_2 x_2 + c_2 v_2) + \gamma_2 u_2
\]
\[
\ldots
\]
\[
x_N = \alpha_N S_N x_{N-1} + \beta_N T_N(\alpha_N T_N x_N + b_N S_N x_N + c_N v_N) + \gamma_N u_N
\]
\[
x_{N+1} = \alpha_{N+1} S_{N+1} x_N + \beta_{N+1} T_{N+1}(\alpha_{N+1} T_{N+1} x_{N+1} + b_{N+1} S_{N+1} x_{N+1} + c_{N+1} v_{N+1}) + \gamma_{N+1} u_{N+1}
\]
\[
x_{N+2} = \alpha_{N+2} S_{N+2} x_{N+1} + \beta_{N+2} T_{N+2}(\alpha_{N+2} T_{N+2} x_{N+2} + b_{N+2} S_{N+2} x_{N+2} + c_{N+2} v_{N+2}) + \gamma_{N+2} u_{N+2}
\]
\[
\ldots
\]
Since for each \( n \geq 1 \), it can be written as \( n = (k - 1)N + i, \quad i \in \{1, 2, \ldots, N\}, \quad k = k(n) \geq 1 \)
is some positive integer and \( k(n) \to \infty \) as \( n \to \infty \). Hence we can written the above table in the following compact form:
\[
x_n = \alpha_n S^{k(n)} x_{n-1} + \beta_n T^{k(n)}_{i(n)}(\alpha_n T^{k(n)}_{i(n)} x_n + b_n S^{k(n)}_{i(n)} x_n + c_n v_n) + \gamma_n u_n, \quad \forall n \geq 1.
\]
Putting \( y_n = a_n T^{k(n)}_{i(n)} x_n + b_n S^{k(n)}_{i(n)} x_n + c_n v_n \), we have the following composite iterative scheme:

\[
\begin{align*}
\{x_n &= \alpha_n x_{n-1} + \beta_n T^{k(n)}_{i(n)} y_n + \gamma_n u_n, \\
y_n &= a_n T^{k(n)}_{i(n)} x_n + b_n S^{k(n)}_{i(n)} x_n + c_n v_n \}
\tag{2.2}
\end{align*}
\]

**Lemma 2.4** [16] Let \( \{r_n\} \), \( \{s_n\} \), \( \{t_n\} \) be sequences of nonnegative real numbers satisfying the inequality

\[ r_{n+1} \leq (1 + s_n) r_n + t_n, \quad n \geq 1. \]

If \( \sum_{n=1}^{\infty} s_n < \infty \) and \( \sum_{n=1}^{\infty} t_n < \infty \), then

(i) \( \lim_{n \to \infty} a_n \) exists,

(ii) In particular, if \( \{a_n\} \) has a subsequence \( \{a_{n_k}\} \) converging to 0, then \( \lim_{n \to \infty} a_n = 0 \).

**Lemma 2.5** [4] Let \( E \) be a uniformly convex Banach space and \( B_D = \{x \in E : \|x\| \leq D\}, \ D > 0 \). Then there exists a continuous strictly increasing and convex function \( g : [0, \infty) \to [0, \infty), \ g(0) = 0 \) such that

\[ \lambda \|x + y + \gamma z\|^2 \leq \lambda \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \lambda \beta g(\|x - y\|), \]

for all \( x, y, z \in B_D \) and \( \lambda, \beta, \gamma \in [0, 1] \) with \( \lambda + \beta + \gamma = 1 \).

### 3 Main result

In this section, we first prove following Lemma, which forms a major part of the proofs of strong convergence theorems.

**Lemma 3.1** Let \( C \) be a nonempty convex subset of normed linear space \( E \). Let \( \{S_i : i \in I'\} \) and \( \{T_i : i \in I'\} \) be two finite families of generalized asymptotically quasi-nonexpansive self-mappings of \( C \) with \( \{h_n\}, \{\lambda_n\} \subset [0, \infty) \). Suppose that \( F = \bigcap_{i=1}^{N} F(S_i) \cap \bigcap_{i=1}^{N} F(T_i) \) is nonempty. Let \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\} \) be real sequences in \((0, 1)\) with \( \alpha_n + \beta_n + \gamma_n = 1 = a_n + b_n + c_n \) for all \( n \geq 1 \) satisfy the following restrictions:

(a) \( 0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1 \),

(b) \( \sum_{n=1}^{\infty} \gamma_n < \infty \), \( \sum_{n=1}^{\infty} h_n < \infty \),

(c) \( \sum_{n=1}^{\infty} \beta_n c_n < \infty \), \( \sum_{n=1}^{\infty} \beta_n \lambda_n < \infty \).

For any \( x_0 \in C \), let the composite implicit iteration process with errors \( \{x_n\} \) be generated by (2.2). Then

(1) \( \lim_{n \to \infty} \|x_n - p\| \) exists for all \( p \in F \),

(2) \( \lim_{n \to \infty} d(x_n, F) \) exists, where \( d(x_n, F) = \inf_{p \in F} \|x_n - p\| \).

**Proof** For given \( p \in F \). Since \( \{u_n\}, \{v_n\} \) are bounded sequences in \( C \). We denote \( M = \max \{\sup_{n \geq 1} \|u_n - p\|, \sup_{n \geq 1} \|v_n - p\|\} \). Thus, \( M < \infty \). Since \( S_{i(n)} \) and \( T_{i(n)} \) is a generalized asymptotically quasi-nonexpansive mapping, it follows from (2.2) that we have

\[
\|y_n - p\| \leq a_n \|T^{k(n)}_{i(n)} x_n - p\| + b_n \|S^{k(n)}_{i(n)} x_n - p\| + c_n \|v_n - p\| \\
\leq a_n [(1 + h_{k(n)})\|x_n - p\| + \lambda_n] + b_n [(1 + h_{k(n)})\|x_n - p\| + \lambda_n] + c_n \|v_n - p\| \\
\leq (1 + h_{k(n)})\|x_n - p\| + \lambda_n + c_n M
\tag{3.1}
\]
It follows from (2.2), (3.1) that
\[ \|x_n - p\| \leq \alpha_n\|S^{k(n)}_{ \ell (n)}x_{n-1} - p\| + \beta_n\|T^{k(n)}_{ \ell (n)}y_n - p\| + \gamma_n\|u_n - p\| \]
\[ \leq \alpha_n[(1 + h_{k(n)})\|x_{n-1} - p\| + \lambda_n] + \beta_n[(1 + h_{k(n)})\|y_n - p\| + \lambda_n] + \gamma_n M \]
\[ \leq \alpha_n(1 + h_{k(n)})\|x_{n-1} - p\| + \beta_n(1 + h_{k(n)})^2\|x_n - p\| \]
\[ + M(1 + h_{k(n)})\beta_n c_n + (1 + h_{k(n)})\beta_n \lambda_n + \gamma_n M + \lambda_n \]
\[ \leq \alpha_n(1 + h_{k(n)})\|x_{n-1} - p\| + (1 - \alpha_n - \gamma_n)(1 + h_{k(n)})\|x_n - p\| + \mu_n \]
\[ \leq \alpha_n(1 + h_{k(n)})\|x_{n-1} - p\| + (1 - \alpha_n)[1 + h_{k(n)}(2 + h_{k(n)})]\|x_n - p\| + \mu_n \]
\[ \leq \alpha_n(1 + h_{k(n)})\|x_{n-1} - p\| + (1 - \alpha_n)\|x_n - p\| + \theta_n\|x_n - p\| + \mu_n \]  \tag{3.2}
where \( \mu_n = M(1 + h_{k(n)})\beta_n c_n + (1 + h_{k(n)})\beta_n \lambda_n + \gamma_n M + \lambda_n \) and \( \theta_n = h_{k(n)}(2 + h_{k(n)}) \). From restrictions (b) and (c) \( \sum_{n=1}^{\infty} \alpha_n < \infty, \sum_{n=1}^{\infty} \beta_n < \infty \). It follows from (3.2) that
\[ \alpha_n\|x_n - p\| \leq \alpha_n[1 + h_{k(n)}]\|x_{n-1} - p\| + \theta_n\|x_n - p\| + \mu_n \]  \tag{3.3}
Because \( 0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1 \), there exist a positive integer \( \eta_0 \) and \( \eta, \eta' \in (0, 1) \) such that
\[ 0 < \eta < \alpha_n < \eta' < 1 \]  \tag{3.4}
for all \( \eta \geq \eta_0 \). From (3.3) and (3.4), we have
\[ \|x_n - p\| \leq [1 + h_{k(n)}]\|x_{n-1} - p\| + \frac{\theta_n}{\eta}\|x_n - p\| + \frac{\mu_n}{\eta}, \quad \forall \eta \geq \eta_0. \]  \tag{3.5}
By transposing (3.5), we have
\[ \frac{\eta - \theta_n}{\eta}\|x_n - p\| \leq [1 + h_{k(n)}]\|x_{n-1} - p\| + \frac{\mu_n}{\eta}, \quad \forall \eta \geq \eta_0. \]  \tag{3.6}
Since \( \lim_{n \to \infty} \theta_n = \lim_{n \to \infty} h_{k(n)}[2 + h_{k(n)}] = 0 \), there exists a positive integer \( \eta_1(\geq \eta_0) \) such that \( \eta - \theta_n > 0 \) and \( \theta_n < \frac{\eta}{2} \) for all \( \eta > \eta_1 \). It follows from (3.6) that
\[ \|x_n - p\| \leq [1 + h_{k(n)}]\frac{\eta}{\eta - \theta_n}\|x_{n-1} - p\| + \frac{\mu_n}{\eta - \theta_n}, \quad \forall \eta \geq \eta_1. \]  \tag{3.7}
Let
\[ 1 + w_n = \frac{\eta}{\eta - \theta_n} = 1 + \frac{\theta_n}{\eta - \theta_n}, \]
then
\[ w_n = \frac{\theta_n}{\eta - \theta_n} < \frac{\theta_n}{\theta_n} < \frac{\theta_n}{\frac{\eta}{2}} < \frac{2\theta_n}{\eta}, \quad \forall \eta > \eta_1. \]
Since \( \sum_{n=1}^{\infty} \theta_n < \infty \), then \( \sum_{n=1}^{\infty} w_n < \infty \). It follows from (3.7) that
\[ \|x_n - p\| \leq (1 + h_{k(n)})(1 + w_n)\|x_{n-1} - p\| + \frac{2\mu_n}{\eta}, \quad \forall \eta \geq \eta_1. \]  \tag{3.8}
By Lemma 2.4 and (3.8), we have that \( \lim_{n \to \infty} \|x_n - p\| \) exists. Taking infimum over \( p \in F \) in (3.8), we have
\[ d(x_n, F) \leq (1 + h_{k(n)})(1 + w_n)d(x_{n-1}, F) + \frac{2\mu_n}{\eta}, \quad \forall \eta \geq \eta_1. \]  \tag{3.9}
Therefore, \( \lim_{n \to \infty} d(x_n, F) \) exists. This completes the proof.

**Lemma 3.2** Let \( C \) be a nonempty convex subset of a real uniformly convex Banach spaces \( E \). Let \( \{S_i : i \in I'\} \) and \( \{T_i : i \in I'\} \) be uniformly continuous and generalized asymptotically quasi-nonexpansive self mappings of \( C \) with sequences \( \{h_n\}, \{\alpha_n\} \subset [0, \infty) \). Now suppose that \( F = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \) is nonempty. Let \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\} \) be real sequences in \( (0, 1) \) with \( \alpha_n + \beta_n + \gamma_n = 1 = a_n + b_n + c_n \) for all \( n \geq 1 \) satisfy the following restrictions:

(a) \( 0 < \lim \inf_{n \to \infty} \alpha_n \leq \lim \sup_{n \to \infty} \alpha_n < 1, \quad 0 < \lim \inf_{n \to \infty} a_n \leq \lim \sup_{n \to \infty} (a_n + c_n) < 1, \)

(b) \( \sum_{i=1}^\infty \gamma_n < \infty, \quad \sum_{i=1}^\infty h_n < \infty, \)

(c) \( \sum_{i=1}^\infty \beta_n c_n < \infty, \quad \sum_{i=1}^\infty \beta_n \lambda_n < \infty, \)

(d) \( \lim_{n \to \infty} c_n = 0. \)

For any \( x_0 \in C \), let the composite implicit iteration process with errors \( \{x_n\} \) be generated by (2.2). Then \( \lim_{n \to \infty} \|x_n - S_{i(n)} x_n\| = 0 \) and \( \lim_{n \to \infty} \|x_n - T_{i(n)} x_n\| = 0 \) for all \( i(n) \in I' \).

**Proof** Let \( p \in F \), by the Lemma 3.1, we have \( \lim_{n \to \infty} \|x_n - p\| \) exists. Now the sequence \( \{x_n - p\} \) is bounded. Since \( \{S_i\}_{i=1}^N \) and \( \{T_i\}_{i=1}^N \) are generalized quasi-nonexpansive mappings, we have that \( \{S_{i(n)} x_n - p\} \) is a bounded sequence for all \( i(n) \in I' \). From (2.2), we have

\[
\|y_n - p\| \leq a_n \|T_{i(n)}^{k(n)} x_n - p\| + b_n \|S_{i(n)}^{k(n)} x_n - p\| + c_n \|v_n - p\|.
\]

Therefore, \( \{y_n - p\} \) is also bounded. We know that \( \{T_{i(n)}^{k(n)} y_n - p\} \) is bounded too. By setting

\[
D = \max \{ \sup_n \|x_n - p\|, \sup_n \|S_{i(n)}^{k(n)} x_n - p\|, \sup_n \|T_{i(n)}^{k(n)} y_n - p\|, M \},
\]

Thus, \( D < \infty \). It follows from Lemma 2.5 and (2.2) that

\[
\|y_n - p\|^2 = \|a_n (T_{i(n)}^{k(n)} x_n - p) + b_n (S_{i(n)}^{k(n)} x_n - p) + c_n (v_n - p)\|^2
\]

\[
\leq a_n \|T_{i(n)}^{k(n)} x_n - p\|^2 + b_n \|S_{i(n)}^{k(n)} x_n - p\|^2 + c_n \|v_n - p\|^2 - a_n b_n g(\|T_{i(n)}^{k(n)} x_n - S_{i(n)}^{k(n)} x_n\|)
\]

\[
\leq a_n [(1 + h_{k(n)}) \|x_n - p\|^2 + \lambda_n^2] + b_n [(1 + h_{k(n)}) \|x_n - p\|^2 + \lambda_n^2]
\]

\[
+ M^2 c_n - a_n b_n g(\|T_{i(n)}^{k(n)} x_n - S_{i(n)}^{k(n)} x_n\|)
\]

\[
\leq (1 + h_{k(n)}) \|x_n - p\|^2 + \lambda_n [\lambda_n + 2(1 + h_{k(n)}) D]
\]

\[
+ M^2 c_n - a_n b_n g(\|T_{i(n)}^{k(n)} x_n - S_{i(n)}^{k(n)} x_n\|)
\]

(3.10)

For some constant \( R_1 > 0 \), it follows from (3.10) that

\[
\|y_n - p\|^2 \leq [1 + h_{k(n)}^2] \|x_n - p\|^2 + (\lambda_n + c_n) R_1 - a_n (1 - a_n - c_n) g(\|T_{i(n)}^{k(n)} x_n - S_{i(n)}^{k(n)} x_n\|)
\]

(3.11)

Similar to (3.10), then we have

\[
\|x_n - p\|^2 \leq \alpha_n \|S_{i(n)}^{k(n)} x_n - p\|^2 + \beta_n \|T_{i(n)}^{k(n)} y_n - p\|^2 + \gamma_n \|u_n - p\|^2 - \alpha_n \beta_n g(\|S_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} y_n\|)
\]

\[
\leq \alpha_n [(1 + h_{k(n)}) \|x_n - p\|^2 + \lambda_n^2] + \beta_n [(1 + h_{k(n)}) \|y_n - p\|^2 + \lambda_n^2]
\]

\[
+ M^2 \gamma_n - \alpha_n \beta_n g(\|S_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} y_n\|)
\]

(3.12)
For some constant \( R_2 > 0 \), it follows from (3.12) that
\[
\| x_n - p \|^2 \leq \alpha_n (1 + h(k(n)))^2 \| x_{n-1} - p \|^2 + 2\alpha_n \lambda_n (1 + h(k(n))) \| x_{n-1} - p \|^2 \\
+ \beta_n (1 + h(k(n)))^2 \| y_n - p \|^2 - \alpha_n \beta_n g(\| S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} y_n \|) \\
+ \alpha_n^2 \lambda_n^2 + M^2 \gamma_n + \beta_n \lambda_n^2 + 2\beta_n \lambda_n (1 + h(k(n))) \| (1 + h(k(n))) D + \lambda_n + c_n M \\
\leq \alpha_n (1 + h(k(n)))^2 \| x_{n-1} - p \|^2 + 2\alpha_n \lambda_n (1 + h(k(n))) \| x_{n-1} - p \|^2 + \beta_n (1 + h(k(n)))^2 \| y_n - p \|^2 \\
+ \beta_n \lambda_n [\lambda_n + 2(1 + h(k(n))) ((1 + h(k(n))) D + \lambda_n + c_n M)] \\
+ M^2 \gamma_n + \alpha_n \lambda_n^2 - \alpha_n \beta_n g(\| S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} y_n \|) \\
\leq \alpha_n (1 + h(k(n)))^2 \| x_{n-1} - p \|^2 + 2\alpha_n \lambda_n (1 + h(k(n))) \| x_{n-1} - p \|^2 + \beta_n (1 + h(k(n)))^2 \| y_n - p \|^2 \\
+ R_2 [\beta_n \lambda_n + \gamma_n] + \alpha_n \lambda_n^2 - \alpha_n \beta_n g(\| S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} y_n \|) \\
(3.13)
\]

Substituting (3.11) into (3.13), for some constant \( R > 0 \), then we get
\[
\| x_n - p \|^2 \leq \alpha_n (1 + h(k(n)))^2 \| x_{n-1} - p \|^2 + 2\alpha_n \lambda_n (1 + h(k(n))) \| x_{n-1} - p \|^2 + \beta_n (1 + h(k(n)))^4 \| x_n - p \|^2 \\
+ (1 + h(k(n)))^2 (\beta_n \lambda_n + \beta_n c_n) R_1 (1 + h(k(n)))^2 \beta_n \alpha_n b_n g(\| T_{i(n)}^{k(n)} x_n - S_{i(n)}^{k(n)} x_n \|) \\
+ R_2 [\beta_n \lambda_n + \gamma_n] + \alpha_n \lambda_n^2 - \alpha_n \beta_n g(\| S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} y_n \|) \\
\leq \alpha_n \| x_{n-1} - p \|^2 + 2\alpha_n \lambda_n \| x_{n-1} - p \| + (1 - \alpha_n) \| x_n - p \|^2 + R(h(k_n) + \beta_n \lambda_n + \beta_n c_n + \gamma_n) \\
- \alpha_n (1 - \alpha_n - \gamma_n) (1 - a_n - c_n) g(\| T_{i(n)}^{k(n)} x_n - S_{i(n)}^{k(n)} x_n \|) \\
+ \alpha_n \lambda_n^2 - \alpha_n (1 - \alpha_n - \gamma_n) g(\| S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} y_n \|) \\
(3.14)
\]

By transposing and simplifying (3.14), then we get
\[
\| x_n - p \|^2 \leq \| x_{n-1} - p \|^2 + (h(k_n) + \beta_n \lambda_n + \beta_n c_n + \gamma_n) \frac{R}{\alpha_n} \\
- \frac{\alpha_n (1 - \alpha_n - \gamma_n)(1 - a_n - c_n)}{\alpha_n} g(\| T_{i(n)}^{k(n)} x_n - S_{i(n)}^{k(n)} x_n \|) \\
- (1 - \alpha_n - \gamma_n) g(\| S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} y_n \|) + \lambda_n^2 \\
(3.15)
\]

It follows from (3.15) that
\[
(1 - \alpha_n - \gamma_n) g(\| S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} y_n \|) \leq \| x_{n-1} - p \|^2 - \| x_n - p \|^2 + \lambda_n^2 \\
+ (h(k_n) + \beta_n \lambda_n + \beta_n c_n + \gamma_n) \frac{R}{\alpha_n} \\
\]

and
\[
\frac{\alpha_n (1 - \alpha_n - \gamma_n)(1 - a_n - c_n)}{\alpha_n} g(\| T_{i(n)}^{k(n)} x_n - S_{i(n)}^{k(n)} x_n \|) \leq \| x_{n-1} - p \|^2 - \| x_n - p \|^2 + \lambda_n^2 \\
+ (h(k_n) + \beta_n \lambda_n + \beta_n c_n + \gamma_n) \frac{R}{\alpha_n} \\
(3.16)
\]

Since \( \sum_{n=1}^{\infty} \gamma_n < \infty \) implies that \( \lim_{n \to \infty} \gamma_n = 0 \), then we have \( \lim \sup_{n \to \infty} (\alpha_n + \gamma_n) = \lim \sup_{n \to \infty} \alpha_n + \lim_{n \to \infty} \gamma_n = \lim \sup_{n \to \infty} \alpha_n \). Since \( 0 < \lim \inf_{n \to \infty} \alpha_n \leq \lim \sup_{n \to \infty} \alpha_n < 1 \), there exist a positive integer \( \eta_0 \) and \( \eta, \eta' \in (0, 1) \) such that
\[
0 < \eta < \alpha_n \leq (\alpha_n + \gamma_n) < \eta' < 1, \quad \forall \eta \geq \eta_0. \\
(3.18)
\]
It follows from (3.16) and (3.18) then we get
\[
(1 - \eta)\|S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} y_n\| \leq \|x_{n-1} - p\|^2 - \|x_n - p\|^2 + (h_k(n) + \beta_n \lambda_n + \beta_n c_n + \gamma_n) \frac{R}{\eta} + \lambda_n^2, \quad \forall \eta \geq \eta_0 \tag{3.19}
\]
By (3.19), \( m \geq \eta_0 \),
\[
\sum_{\eta=\eta_0}^{m} g\|S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} y_n\| \leq \frac{1}{1 - \eta'} \left[ \sum_{\eta=\eta_0}^{m} (\|x_{n-1} - p\|^2 - \|x_n - p\|^2) + \sum_{\eta=\eta_0}^{m} (h_k(n) + \beta_n \lambda_n + \beta_n c_n + \gamma_n) \frac{R}{\eta} \right] + \sum_{\eta=\eta_0}^{m} \lambda_n^2 \tag{3.20}
\]
Let \( m \to \infty \) in (3.20) then we get \( \sum_{\eta=\eta_0}^{m} g\|S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} y_n\| \to \infty \),
therefore \( \lim_{n \to \infty} g\|S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} y_n\| = 0 \).
Since \( g \) is strictly increasing and continuous with \( g(0) = 0 \), then we have
\[
\lim_{n \to \infty} \|S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} y_n\| = 0. \tag{3.21}
\]
By similar method, together with (3.17), it can be shown that
\[
\lim_{n \to \infty} \|T_{i(n)}^{k(n)} x_n - S_{i(n)}^{k(n)} x_n\| = 0. \tag{3.22}
\]
It follows from (2.2) that
\[
\|x_n - S_{i(n)}^{k(n)} x_{n-1}\| = \|\alpha_n S_{i(n)}^{k(n)} x_{n-1} + \beta_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n - S_{i(n)}^{k(n)} x_{n-1}\|
\leq \beta_n \|T_{i(n)}^{k(n)} y_n - S_{i(n)}^{k(n)} x_{n-1}\| + \gamma_n \|u_n - S_{i(n)}^{k(n)} x_{n-1}\|
\]
Here (3.21) and \( \lim_{n \to \infty} \gamma_n = 0 \) implies that
\[
\lim_{n \to \infty} \|x_n - S_{i(n)}^{k(n)} x_{n-1}\| = 0. \tag{3.23}
\]
Again it follows from (2.2) that
\[
x_n - x_{n-1} = \alpha_n (S_{i(n)}^{k(n)} x_{n-1} - x_{n-1}) + \beta_n (T_{i(n)}^{k(n)} y_n - x_{n-1}) + \gamma_n (u_n - x_{n-1}),
\]
this together with (3.21), (3.23) and \( \lim_{n \to \infty} \gamma_n = 0 \) implies
\[
\lim_{n \to \infty} \|x_n - x_{n-1}\| = 0. \tag{3.24}
\]
Also \( \lim_{n \to \infty} \|x_n - x_{n+l}\| = 0 \) for all \( l \in I' \).
It follows from (2.2) that
\[
\|x_n - T_{i(n)}^{k(n)} y_n\| = \|\alpha_n S_{i(n)}^{k(n)} x_{n-1} + \beta_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n - T_{i(n)}^{k(n)} y_n\|
\leq \alpha_n \|S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} y_n\| + \gamma_n \|u_n - T_{i(n)}^{k(n)} y_n\|
\]
Here (3.21) and \( \lim_{n \to \infty} \gamma_n = 0 \) implies that
\[
\lim_{n \to \infty} \|x_n - T^{(n)}_{i(n)} y_n\| = 0. \tag{3.25}
\]

It follows from (2.2) that \( y_n - T^{(n)}_{i(n)} x_n = b_n (S^{(n)}_{i(n)} x_n - T^{(n)}_{i(n)} x_n) + c_n (v_n - T^{(n)}_{i(n)} x_n) \), this together with (3.22) and \( \lim_{n \to \infty} c_n = 0 \) implies
\[
\lim_{n \to \infty} \|y_n - T^{(n)}_{i(n)} x_n\| = 0. \tag{3.26}
\]

Similarly it follows from (2.2) that \( y_n - S^{(n)}_{i(n)} x_n = a_n (T^{(n)}_{i(n)} x_n - S^{(n)}_{i(n)} x_n) + c_n (v_n - S^{(n)}_{i(n)} x_n) \), this together with (3.22) and \( \lim_{n \to \infty} c_n = 0 \) implies
\[
\lim_{n \to \infty} \|y_n - S^{(n)}_{i(n)} x_n\| = 0. \tag{3.27}
\]

Again it follows from (2.2) that
\[
y_n - T^{(n)}_{i(n)} y_n = a_n (T^{(n)}_{i(n)} x_n - T^{(n)}_{i(n)} y_n) + b_n (S^{(n)}_{i(n)} x_n - T^{(n)}_{i(n)} y_n) + c_n (v_n - T^{(n)}_{i(n)} y_n),
\]
this together with (3.26), (3.27) and \( \lim_{n \to \infty} c_n = 0 \) implies
\[
\lim_{n \to \infty} \|y_n - T^{(n)}_{i(n)} y_n\| = 0. \tag{3.28}
\]

Again we observe that by (2.2), \( y_n - S_n = a_n (T^{(n)}_{i(n)} x_n - x_n) + b_n (S^{(n)}_{i(n)} x_n - x_n) + c_n (v_n - x_n) \), this together with (3.22), (3.25), (3.27), (3.28) and \( \lim_{n \to \infty} c_n = 0 \) implies
\[
\lim_{n \to \infty} \|y_n - x_n\| = 0. \tag{3.29}
\]

Since \( \|S^{(n)}_{i(n)} x_n - x_{n-1}\| \leq \|S^{(n)}_{i(n)} x_n - y_n\| + \|y_n - x_{n-1}\| \), together with (3.24), (3.27) and (3.29) we have
\[
\lim_{n \to \infty} \|S^{(n)}_{i(n)} x_n - x_{n-1}\| = 0. \tag{3.30}
\]

Clearly, \( n \equiv (n - N) (mod N) \) for \( n > N \). So \( S_n = S_{n - N} = S_{i(n)} \) and we have that
\[
\|x_{n-1} - S_n x_n\| \leq \|x_{n-1} - S^{(n)}_{i(n)} x_n\| + \|S^{(n)}_{i(n)} x_n - S_n x_n\|
\leq \|x_{n-1} - S^{(n)}_{i(n)} x_n\| + \|S^{(n)}_{i(n)} x_n - S^{(n)}_{i(n)-N} x_{n-N}\|
+ \|S^{(n)}_{i(n)-N} x_{n-N} - S_{n-N} x_{(n-N)-1}\| + \|S_{n-N} x_{(n-N)-1} - S_n x_n\|
\leq \|x_{n-1} - S^{(n)}_{i(n)} x_n\| + \|S^{(n)}_{i(n)} x_n - S^{(n)}_{i(n)-N} x_{n-N}\|
+ \|S^{(n)}_{i(n)-N} x_{n-N} - S_{n} x_{(n-N)-1}\| + \|S_{n} x_{(n-N)-1} - S_n x_n\|
\]
Since each \( S_{i(n)} \) is a uniformly continuous, then we can deduce that
\[
\lim_{n \to \infty} \|x_{n-1} - S_n x_n\| = 0.
\]

Thus, we get for \( i(n) \in I' \),
\[
\|x_n - S_{n+i(n)} x_n\| \leq \|x_n - x_{n+i(n)}\| + \|x_{n+i(n)} - S_{n+i(n)} x_{n+i(n)}\|
+ \|S_{n+i(n)} x_{n+i(n)} - S_{n+i(n)} x_n\| \to 0, \quad \text{as} \ n \to \infty.
\]
Since any subsequence of a convergent sequence to the same limit, we have that

\[ \lim_{n \to \infty} \| x_n - S_{i(n)} x_n \| = 0, \quad \forall \ i(n) \in I'. \]

Notice that \( T_i \) is uniformly continuous, then we obtain

\[ \| T_{i(n)}^{k(n)} y_n - T_{i(n)}^{k(n)} x_n \| \leq \| T_{i(n)}^{k(n)} y_n - y_n \| + \| y_n - T_{i(n)}^{k(n)} x_n \|, \]

this together with (3.26) and (3.28) implies

\[ \lim_{n \to \infty} \| T_{i(n)}^{k(n)} y_n - T_{i(n)}^{k(n)} x_n \| = 0. \tag{3.31} \]

Since \( \| T_{i(n)}^{k(n)} x_n - x_n \| \leq \| T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} y_n \| + \| T_{i(n)}^{k(n)} y_n - x_n \| \), this together with (3.25) and (3.31) implies

\[ \lim_{n \to \infty} \| T_{i(n)}^{k(n)} x_n - x_n \| = 0. \tag{3.32} \]

Notice that \( \| T_{i(n)}^{k(n)} x_n - x_{n-1} \| \leq \| T_{i(n)}^{k(n)} x_n - x_n \| + \| x_n - x_{n-1} \| \), by (3.24) and (3.32) implies

\[ \lim_{n \to \infty} \| T_{i(n)}^{k(n)} x_n - x_{n-1} \| = 0. \]

By similar method, it can be shown that \( \lim \| x_n - T_{n+i(n)} x_n \| = 0 \) for all \( i(n) \in I' \). This implies that \( \lim_{n \to \infty} \| x_n - T_{i(n)} x_n \| = 0 \) for all \( i(n) \in I' \). This completes the proof.

Now we state and prove our main result for generalized asymptotically quasi-nonexpansive mappings in a Banach space.

**Theorem 3.3** Let \( C \) be a nonempty convex subset of a real Banach space \( E \). Let \( \{ S_i : i \in I' \} \) and \( \{ T_i : i \in I' \} \) be two uniformly continuous and generalized asymptotically quasi-nonexpansive self-mappings of \( C \) with \( \{ h_n \} , \{ \lambda_n \} \subset [0, \infty) \). Suppose that \( F = \bigcap_{i=1}^{N} F(S_i) \cap \bigcap_{i=1}^{N} F(T_i) \) is nonempty. Let \( \{ \alpha_n \} , \{ \beta_n \} , \{ \gamma_n \} , \{ a_n \} , \{ b_n \} , \{ c_n \} \) be real sequences in \((0,1)\) with \( \alpha_n + \beta_n + \gamma_n = 1 = a_n + b_n + c_n \) for all \( n \geq 1 \) satisfying the following restrictions:

(a) \( 0 < \lim \inf_{n \to \infty} \alpha_n \leq \lim \sup_{n \to \infty} \alpha_n < 1 \),

(b) \( \sum_{n=1}^{\infty} \gamma_n < \infty \), \( \sum_{n=1}^{\infty} h_n < \infty \),

(c) \( \sum_{n=1}^{\infty} \beta_n c_n < \infty \), \( \sum_{n=1}^{\infty} \beta_n \lambda_n < \infty \).

For any initial point \( x_0 \in C \), let the composite implicit iteration process with errors, \( \{ x_n \} \) be generated by (2.2). Then \( \{ x_n \} \) converges strongly to a common fixed point of the mapping \( \{ S_i : i \in I' \} \) and \( \{ T_i : i \in I' \} \) if and only if \( \lim \inf_{n \to \infty} d(x_n, F) = 0 \), where \( d(x_n, F) = \inf_{y \in F} \| x_n - y \| \), for all \( n \geq 1 \).

**Proof** The necessity is obvious. Indeed, if \( x_n \to x^* \in F(n \to \infty) \), then

\[ d(x_n, F) = \inf_{x^* \in F} d(x_n, x^*) \leq \| x_n - x^* \| \to 0, \quad (n \to \infty), \]

i.e. \( \lim_{n \to \infty} d(x_n, F) = 0 \). Now we prove sufficiency. Let \( \lim \inf_{n \to \infty} d(x_n, F) = 0 \), this together with Lemma 2.4 and (3.9), we have \( \lim_{n \to \infty} d(x_n, F) = 0 \). Now next we claim that \( \{ x_n \} \) is a cauchy
sequence. By (3.8) for any $p \in F$ then we have
\[
\|x_{n+m} - p\| \leq (1 + h_{k(n)}) \prod_{j=n}^{n+m-1} (1 + w_j) \|x_n - p\| + \frac{2}{\eta} \prod_{j=n}^{n+m-1} (1 + w_j) \left( \sum_{j=n}^{n+m-1} \mu_j \right) \\
\leq (1 + h_{k(n)}) \exp \left( \sum_{j=n}^{n+m-1} w_j \right) \|x_n - p\| + \frac{2}{\eta} \exp \left( \sum_{j=n}^{n+m-1} w_j \right) \left( \sum_{j=n}^{n+m-1} \mu_j \right) \\
\leq M_1 \left( 1 + h_{k(n)} \|x_n - p\| + \sum_{j=1}^{\infty} \mu_j \right)
\]
for all natural numbers $m, n$, where $M_1 = \frac{2}{\eta} \exp \left( \sum_{j=1}^{\infty} w_j \right) < \infty$. Since $\lim_{n \to \infty} d(x_n, F) = 0$, given any $\epsilon > 0$, there exists a natural number $N_1$ such that $d(x_n, F) < \epsilon/4M_1(1 + h_{k(n)})$ and $\sum_{n=N_1}^{\infty} \mu_n < \epsilon/4M_1(1 + h_{k(n)})$ for all $n \geq N_1$. Thus, we can find $x^* \in F$ such that $\|x_{N_1} - x^*\| \leq \epsilon/4M_1(1 + h_{k(n)})$. Hence, for all $n \geq N_1$ and $m \geq 1$, then we get
\[
\|x_{n+m} - x_n\| \leq \|x_{n+m} - x^*\| + \|x_n - x^*\| \\
\leq 2M_1(1 + h_{k(n)}) \left( \|x_{N_1} - x^*\| + \sum_{j=1}^{\infty} \mu_j \right) \\
\leq 2M_1(1 + h_{k(n)}) \left( \frac{\epsilon}{4M_1(1 + h_{k(n)})} + \frac{\epsilon}{4M_1(1 + h_{k(n)})} \right) = \epsilon.
\]
This show that $\{x_n\}$ is a cauchy sequence. Thus $\lim_{n \to \infty} x_n$ is exists. Let $\lim_{n \to \infty} x_n = p^*$. Again we show that $p^*$ is a common fixed point of the mapping $\{S_i : i \in I'\}$ and $\{T_i : i \in I'\}$. i. e., we show that $p^* \in F$. Let for contradiction that $p^* \in F^c$. Since $\{S_i : i \in I'\}$ and $\{T_i : i \in I'\}$, are continuous, $F$ is a closed subset of $E$. Therefore, we have $d(p^*, F) > 0$. Notice that for all $p \in F$, then we have $\|p^* - p\| \leq \|p^* - x_n\| + \|x_n - p\|$ is implies that
\[
d(p^*, F) \leq \|x_n - p^*\| + d(x_n, F).
\]
as $n \to \infty$ we get $d(p^*, F) = 0$ which contradicts $d(p^*, F) > 0$. Thus, $p^*$ is a common fixed point of $\{S_i\}_{i=1}^{N}$ and $\{T_i\}_{i=1}^{N}$. This completes the proof.

If in Theorem 3.3, following results:

**Corollary 3.4** Let $C$ be a nonempty convex subset of a real Banach space $E$. Let $\{S_i : i \in I'\}$ and $\{T_i : i \in I'\}$ be two uniformly continuous and generalized asymptotically quasi-nonexpansive self mappings of $C$ with $\{h_n\}$, $\{\lambda_n\} \subset (0, \infty)$. Suppose that $F = \bigcap_{i=1}^{N} F(S_i) \cap \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1 = a_n + b_n + c_n$ for all $n \geq 1$ satisfying the following restrictions:

(a) $0 < \lim \inf_{n \to \infty} \alpha_n \leq \lim \sup_{n \to \infty} \alpha_n < 1$,
(b) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} h_n < \infty$,
(c) $\sum_{n=1}^{\infty} \beta_n c_n < \infty$, $\sum_{n=1}^{\infty} \beta_n \lambda_n < \infty$.

For any initial point $x_0 \in C$, let the composite implicit iteration process with errors, $\{x_n\}$ be generated by (2.2). Then $\{x_n\}$ converges strongly to a common fixed point of the mapping $\{S_i : i \in I'\}$ and $\{T_i : i \in I'\}$ if and only if there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly to $p \in F$. 
Corollary 3.5 Let $C$ be a nonempty convex subset of a real Banach space $E$. Let $\{S_i : i \in I\}$ and $\{T_i : i \in I'\}$ be two uniformly continuous and asymptotically quasi-nonexpansive self-mappings of $C$ with $\{h_n\} \subset [0, \infty)$. Suppose that $F = \bigcap_{i=1}^{N} F(S_i) \cap \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{a_n\}, \{b_n\}, \{c_n\}$ be real sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1 = a_n + b_n + c_n$ for all $n \geq 1$ satisfying the following restrictions:

1. $0 < \lim \inf_{n \to \infty} \alpha_n \leq \lim \sup_{n \to \infty} \alpha_n < 1$,
2. $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} h_n < \infty$,
3. $\sum_{n=1}^{\infty} \beta_n < \infty, \sum_{n=1}^{\infty} \beta_n c_n < \infty$.

For any initial point $x_0 \in C$, let the composite implicit iteration process with errors, $\{x_n\}$ be generated by (2.2). Then $\{x_n\}$ converges strongly to a common fixed point of the mapping $\{S_i : i \in I\}$ and $\{T_i : i \in I'\}$ if and only if $\lim \inf_{n \to \infty} d(x_n, F) = 0$.

Proof: This corollary proof follows from Theorem 3.3 with $\lambda_n = 0$ for all $n \geq 1$.

Theorem 3.6 Let $C$ be a nonempty convex subset of a real Banach space $E$. Let $\{S_i : i \in I\}$ and $\{T_i : i \in I'\}$ be two uniformly continuous and generalized asymptotically quasi-nonexpansive self-mappings of $C$ with $\{h_n\}, \{\lambda_n\} \subset [0, \infty)$. Suppose that $F = \bigcap_{i=1}^{N} F(S_i) \cap \bigcap_{i=1}^{N} F(T_i)$ is nonempty and one of mapping in $\{S_i : i \in I\}$ and $\{T_i : i \in I'\}$ is semi compact. Let $\{a_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\}$ be real sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1 = a_n + b_n + c_n$ for all $n \geq 1$ satisfying the following restrictions:

1. $0 < \lim \inf_{n \to \infty} \alpha_n \leq \lim \sup_{n \to \infty} \alpha_n < 1$,
2. $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} h_n < \infty$,
3. $\sum_{n=1}^{\infty} \beta_n c_n < \infty, \sum_{n=1}^{\infty} \beta_n \lambda_n < \infty$.
4. $\lim_{n \to \infty} c_n = 0$.

For any initial point $x_0 \in C$, let the composite implicit iteration process with errors, $\{x_n\}$ be generated by (2.2). Then $\{x_n\}$ converges strongly to a common fixed point of the mapping $\{S_i : i \in I\}$ and $\{T_i : i \in I'\}$.

Proof Suppose that $S_{i_0}$ and $T_{j_0}$ are semi compact for some $i_0, j_0$. It follows from Lemma 3.2 that there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to p \in C$ and $\lim_{j \to \infty} ||S_{i_0} x_{n_j} - x_{n_j}|| = 0 = \lim_{n \to \infty} ||T_{j_0} x_{n_j} - x_{n_j}||$. Thus by Lemma 3.2, we have $\lim_{j \to \infty} ||x_{n_j} - T_i x_{n_j}|| = 0$. Therefore, we have that

$$\|p - T_i p\| \leq \|p - x_{n_j}\| + \|x_{n_j} - T_i x_{n_j}\| + \|T_i x_{n_j} - T_i p\| \to 0, \quad j \to \infty, \quad for \quad i \in I'.$$

This implies $p \in \bigcap_{i=1}^{N} F(T_i)$. Similarly we can prove that $p \in \bigcap_{i=1}^{N} F(S_i)$. By Theorem 3.3 (b), we get that $x_n \to p$ as $n \to \infty$. This is completes the proof.

4 Conclusion

In this paper, we propose a new composite implicit iterative process with errors which enable us to discuss strong convergence problem to a common fixed point for two finite families of generalized asymptotically quasi-nonexpansive mappings in framework of Banach space. Our results extend recent results of [1, 2, 3, 10, 15, 17].

References


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