THE TOPOLOGY OF GENERALIZED FUZZY METRIC SPACES AND VECTOR IMAGE FILTERING

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Abstract: Khan [12] generalized the concept of Fuzzy metric space (In the sense of George and Veeramani) and introduced the notion of Generalized fuzzy n-metric spaces. In this paper, we further investigate the properties of these generalized fuzzy metric spaces and extend the Banach Fixed point theorem in this new framework. We also propose a vector image filter based on generalized fuzzy metrics.

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1. INTRODUCTION

Kramosil and Michalek [3] introduced the notion of Fuzzy metric space by generalizing the concept of probabilistic metric space introduced by K. Menger [22]. George and Veeramani ([4],[5],[6]) used the concept of continuous t-norms to modify this definition of fuzzy metric space and showed that the topology for their new definition is Hausdorff. These fuzzy spaces have important applications in image filtering ([12],[16]) and quantum particle physics [26]. In 1988, Grabiec [7] first defined the Banach contraction in a fuzzy metric space and extended fixed point theorems of Banach and Edelstein to fuzzy metric spaces. Following this approach, Vasuki [27] generalized the Grabiec’s fuzzy Banach contraction theorem and proved a common fixed point theorem for a sequence of mappings in a fuzzy metric space. Sharma [24] also extended some known results of fixed point theory for compatible mappings in fuzzy metric spaces. In 2002, Gregori and Sapena [8] introduced the notion of fuzzy contractive mapping and proved some fixed point theorems for complete fuzzy metric spaces in the sense of George and Veeramani. Mihet [15] proposed a fuzzy Banach theorem for (weak) B-contraction in M-complete fuzzy metric spaces. There are several generalizations of fuzzy metric spaces (e.g. [23],[35]) for more than two variables. Recently Khan [12] generalized and studied the concept of Generalized fuzzy n-metric space by combining the definition given by George and Veeramani([4],[6]) with that of generalized n-metric space ([10],[11]).

In this paper, we further investigate the properties of generalized fuzzy n-metric spaces and prove the famous Banach fixed point theorem in this new framework.

Fuzzy metrics are found to be useful in colour image filtering techniques ([3],[12],[16]). Morillas et al [16] replaced the classical metrics by fuzzy metrics (in the sense of George and Veeramani) in constructing a variant of Vector Median Filter (VMF). Recently Khan ([24]) proposed a more generalized version of these filters using generalized fuzzy n-metric spaces. In this paper, we propose a vector filter based on Khan’s proposal for colour image processing.
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2. Introduction

Definition 2.1. [23] A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous $t$-norm if it satisfies the following conditions:

1. $*$ is associative and commutative;
2. $*$ is continuous;
3. $a * 1 = a$ for all $a \in [0, 1]$;
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$

for each $a, b, c, d \in [0, 1]$. The examples of continuous $t$-norm are $a * b = ab$ and $a * b = \min(a, b)$.

Definition 2.2. (George and Veeramani) [4] A Fuzzy Metric Space is a triple $(X, M, *)$ where $X$ is a nonempty set, $*$ is a continuous $t$-norm and $M : X \times X \times (0, \infty) \rightarrow [0, 1]$ is a mapping (called fuzzy metric) which satisfies the following properties: for every $x, y, z \in X$ and $s, t > 0$

[FM 1] $M(x, y, t) > 0$;
[FM 2] $M(x, y, t) = 1$ if and only if $x = y$;
[FM 3] $M(x, y, t) = M(y, x, t)$;
[FM 4] $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$
[FM 5] $M(x, y, *) : (0, \infty) \rightarrow [0, 1]$ is continuous.

Then $M$ is called a fuzzy metric on $X$ and $M(x, y, t)$ denotes the degree of nearness between $x$ and $y$ with respect to $t$.

From now on, by a fuzzy metric $M(x, y, t)$ we always mean a fuzzy metric in the sense of George and Veeramani.

Example 2.3. Let $X$ be a non-empty set and $d$ is a metric on $X$. Denote $a * b = a b$ for all $a, b \in [0, 1]$. For each $t > 0$, define

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, *)$ is a fuzzy metric space. We call this fuzzy metric $M_d$ induced by the metric $d$ the standard fuzzy metric.

We call a topological space $(X, \tau)$ fuzzy metrizable if there exists a fuzzy metric $M$ on $X$ such that $\tau = \tau_M$. George and Veeramani ([5], [6]) showed that every fuzzy metric $M$ on $X$ generates a topology $\tau_M$ on $X$. The family of open sets $\{B_M(x, r, t) : x \in X, \ 0 < r < 1, \ t > 0\}$ forms a base for this topology, where $B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ for every $r, 0 < r < 1$ and $t > 0$. They also proved that this topological space is first countable and Hausdorff. Also for a metric space $(X, d)$, the topology generated by $d$ coincides with the topology $\tau_M$ generated by the standard fuzzy metric $M_d$ thereby indicating that every metrizable topological space is fuzzy metrizable. Gregori and Romaguera [9] proved that the collection $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity $\mathcal{U}_M$ compatible with $\tau_M$, where $U_n = \{(x, y) : M(x, y, \frac{1}{n}) > 1 - 1/n\}$ for all $n \in \mathbb{N}$. Hence the topological space $(X, \tau_M)$ is metrizable.

Definition 2.4. [12] A 3-tuple $(X, F_n, *)$ is called Generalized Fuzzy $n$-metric space if $X$ is an arbitrary (non-empty) set, $*$ is a continuous $t$-norm, and $F_n$ is a fuzzy set on $X^n \times (0, \infty)$ satisfying the following conditions for each $x_1, x_2, \ldots, x_n \in X$ and $t, s > 0$: 

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Proposition 2.9. Let \((X, F_n, \ast)\) be a generalized fuzzy n-metric space. Then for \(x, y \in X\) and \(t > 0\), we have

\[
F_n(x, y, y, \ldots, y) = \frac{t}{t + G_n(x, x, \ldots, x)}
\]

Definition 2.7. \([12]\) Let \((X, F_n, \ast)\) be a generalized fuzzy n-metric space. A subset \(A\) of \(X\) is said to be \(F\)-bounded if there exist \(t > 0\) and \(r \in (0, 1)\) such that

\[
F_n(x_1, x_2, \ldots, x_n, t) < 1 - r \quad \text{for all} \quad x_1, x_2, \ldots, x_n \in A
\]

Definition 2.8. \([12]\) A generalized fuzzy n-metric \((F_n, \ast)\) on \(X\) is said to be stationary if \(F_n\) does not depend on \(t\), i.e., for each \(x_1, x_2, \ldots, x_n \in X\) the function \(F_n(x_1, x_2, \ldots, x_n, t)\) is constant.

Example 2.5. Let \((X, G_n)\) be an Generalized n-metric space. Denote \(a \ast b = a \cdot b\) for all \(a, b \in [0, 1]\). For each \(t > 0\), define

\[
F_n(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + G_n(x_1, x_2, \ldots, x_n)}
\]

for all \(x_1, x_2, \ldots, x_n \in \mathbb{R}\). Then \((X, F_n, \ast)\) is a Generalized fuzzy n-metric space.

Proposition 2.10. \([12]\) Let \((X, F_n, \ast)\) be a Generalized fuzzy n-metric space. For \(t > 0\), the open ball \(B_F(x_0, r, t)\) with center \(x_0\) and radius \(0 < r < 1\) is defined by

\[
B_F(x_0, r, t) = \{ y \in X : F_n(x_0, y, y, \ldots, y, t) > 1 - r \}
\]

Definition 2.11. \([12]\) A subset \(A\) of \(X\) is called an open set if for each \(x \in A\) there exist \(t > 0\) and \(0 < r < 1\) such that \(B_F(x, r, t) \subset A\).

3. MAIN RESULTS

Proposition 3.1. Let \((X, F_n, \ast)\) be a generalized fuzzy n-metric space.

If \(F_n(x_1, x_2, \ldots, x_n, T) > 1 - r\) for \(x_1, x_2, \ldots, x_n \in X\), \(T > 0\), \(0 < r < 1\), We can find a \(t_0\), \(0 < t_0 < T\) such that \(F_n(x_1, x_2, \ldots, x_n, t_0) > 1 - r\).
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Proof. Let \( \bar{t} = \text{Sup} \{ t > 0 : F_n(x_1, x_2, ..., x_n, t) \leq 1 - r \} \), then \( \bar{t} > 0 \). Since \( F_n(x_1, x_2, ..., x_n, t) \) is nondecreasing with respect to \( t \) \[12\], hence

\[ F_n(x_1, x_2, ..., x_n, t) > 1 - r \text{ for all } t > \bar{t} \]

Therefore \( T > \bar{t} \) and there exists \( t_0 > 0 \) such that \( \bar{t} < t_0 < T \) and \( F_n(x_1, x_2, ..., x_n, t_0) > 1 - r \). \( \square \)

Proposition 3.2. Every open ball in a Generalized fuzzy \( n \)-metric space is an open set.

Proof. Let \((X, F_n, *)\) be a generalized fuzzy \( n \)-metric space. Consider an open ball \( B_F(x, r, t) \). Let \( y \in B_F(x, r, t) \), hence we have \( F_n(x, y, ..., y, t) > 1 - r \). We shall show that there exists \( R > 0 \) and \( T > 0 \) such that

\[ B_F(y, R, T) \subset B_F(x, r, t) \]

By proposition 3.1 there exists \( t_0, 0 < t_0 < t \), such that \( F_n(x, y, ..., y, t_0) > 1 - r \). Let \( F_n(x, y, ..., y, t_0) = r_1 \). Then there exists \( s, 0 < s < 1 \) such that \( r_1 > 1 - s > 1 - r \). Now for a given \( r_1 \) and \( s \) we can find \( r_2, 0 < r_2 < 1 \), such that

\[ r_1 * r_2 \geq 1 - s \]

Let \( R = 1 - r_2 \) and \( T = t - t_0 \). Then \( z \in B_F(y, R, T) \) implies that \( F_n(y, z, ..., z, t - t_0) > r_2 \). Thus

\[ F_n(x, z, ..., z, t) \geq F_n(x, y, ..., y, t_0) * F_n(y, z, ..., z, t - t_0) \]
\[ > r_1 * r_2 \]
\[ \geq 1 - s \]
\[ > 1 - r \]

Thus \( z \in B_F(x, r, t) \) and hence \( B_F(y, R, T) \subset B_F(x, r, t) \). \( \square \)

Proposition 3.3. Let \((X, F_n, *)\) be a generalized fuzzy \( n \)-metric space. Let \( B_F(x, r_1, t) \) and \( B_F(x, r_2, t) \) be open balls with the same center \( x \in X \) and with radius \( 0 < r_1 < 1 \) and \( 0 < r_2 < 1 \) respectively. Then either we have \( B_F(x, r_1, t) \subseteq B_F(x, r_2, t) \) or \( B_F(x, r_2, t) \subseteq B_F(x, r_1, t) \).

Proof. For \( r_1 = r_2 \), the result is obvious. Suppose that \( r_1 \neq r_2 \). Now there are two possibilities viz. \( 0 < r_1 < r_2 < 1 \) and \( 0 < r_2 < r_1 < 1 \). For the case \( 0 < r_1 < r_2 < 1 \), We have \( 1 - r_2 < 1 - r_1 \). Let \( y \in B_F(x, r_1, t) \), then

\[ F_n(x, y, ..., y, t) > 1 - r_1 > 1 - r_2 \]

Hence \( y \in B_F(x, r_2, t) \) implying that \( B_F(x, r_1, t) \subseteq B_F(x, r_2, t) \). A similar argument can be given for the case \( 0 < r_2 < r_1 < 1 \). \( \square \)

Proposition 3.4. Let \((X, F_m, *)\) be a generalized fuzzy \( m \)-metric space. Then \((X, \tau_F)\) is first countable.

Proof. Let \( t > 0 \) and \( x \in X \). We will show that

\[ H_x = \{ B_F(x, \frac{1}{n}, \frac{t}{n}) : n \in \mathbb{N} \} \]

is a local base for \( x \in X \).

Let \( U \in \tau_F \) and \( x \in U \). Since \( U \) is open, there exists \( 0 < r < 1, t > 0 \) such that

\[ B_F(x, r, t) \subset U \]

(3.1)
Choose \( n \in \mathbb{N} \) such that \( r > \frac{1}{n} \). Let \( z \in B_{F}(x, \frac{1}{n}, \frac{r}{n}) \), then
\[
F_{m}(x, z, z, \ldots, z, \frac{r}{n}) > 1 - \frac{1}{n} > 1 - r
\]
\[
\Rightarrow F_{m}(x, z, z, \ldots, z, t) \geq F_{m}(x, z, z, \ldots, z, \frac{r}{n}) > 1 - r
\]
\[
\Rightarrow z \in B_{F}(x, r, t)
\]

Therefore \( B_{F}(x, \frac{1}{n}, \frac{r}{n}) \subseteq B_{F}(x, r, t) \subseteq U \). Hence \( H_{z} \) is a countable local base for \( x \), i.e. \((X, \tau_{F})\) is a first countable topological space.

**Definition 3.5.** Let \((X, F_{n}, \ast)\) be Generalized fuzzy \( n \)-metric space and \( A \subseteq X \). Let \( \Lambda \) be an index set. A collection \( \{G_{\alpha} : \alpha \in \Lambda\} \) of open sets in \( X \) is called an open cover of \( A \) if \( A \subseteq \{G_{\alpha} : \alpha \in \Lambda\} \).

**Definition 3.6.** A subset \( A \) of a Generalized fuzzy \( n \)-metric space \((X, F_{n}, \ast)\) is said to be compact if every open cover \( G \) of \( A \) has a finite subcover.

**Proposition 3.7.** Every compact subset \( A \) of a generalized fuzzy \( n \)-metric space \( X \) is \( F \)-bounded.

**Proof.** Let \((X, F_{n}, \ast)\) be Generalized fuzzy \( n \)-metric space and \( A \) be a compact subset of \( X \). For fixed values of \( t > 0 \) and \( 0 < r < 1 \), the collection \( \{B_{F}(x, r, t) : x \in A\} \) is an open cover of \( A \). Since \( A \) is compact, there exists \( x_{1}, x_{2}, \ldots, x_{k} \in A \) such that
\[
A \subseteq \bigcup_{i=1}^{k} B_{F}(x_{i}, r, t)
\]

Let \( z_{1}, z_{2}, \ldots, z_{n} \in A \). Then there exists a subset \( \{x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{n}}\} \) of \( \{x_{i} : i = 1, 2, \ldots, k\} \), where \( x_{i_{p}} \)'s are not necessarily distinct, such that \( z_{p} \in B_{F}(x_{i_{p}}, r, t) \).

Hence we have
\[
F_{n}(x_{i_{p}}, z_{p}, \ldots, z_{p}, t) > 1 - r
\]

Therefore on using proposition 2.9 we have
\[
F_{n}(z_{p}, x_{i_{p}}, \ldots, x_{i_{p}}, (n-1)t) \geq F_{n}(x_{i_{p}}, z_{p}, \ldots, z_{p}, t)[n-1] > (1-r)^{n-1} \tag{3.2}
\]

Now using [M 5], we have
\[
F_{n}(z_{1}, z_{2}, \ldots, z_{n}, (n^{2}-1)t) \geq F_{n}(z_{1}, x_{i_{1}}, \ldots, x_{i_{k}}, (n-1)t) * F_{n}(z_{2}, x_{i_{2}}, \ldots, x_{i_{k}}, (n-1)t) * \ldots * F_{n}(z_{n}, x_{i_{n}}, \ldots, x_{i_{k}}, (n-1)t) * F_{n}(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{n}}, (n-1)t) \tag{3.3}
\]

Let \( r_{1} = \min\{F_{n}(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{n}}, t) : 1 \leq i_{1}, i_{2}, \ldots, i_{n} \leq k\} \), then we have
\[
F_{n}(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{n}}, (n-1)t) \geq F_{n}(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{n}}, t) \geq r_{1} \tag{3.4}
\]

In view of relations 3.2, 3.3 and 3.4. We have
\[
F_{n}(z_{1}, z_{2}, \ldots, z_{n}, (n^{2}-1)t) > (1-r)^{n-1} * (1-r)^{n-1} * \ldots * (1-r)^{n-1} * r_{1}
\]

Let \((n^{2}-1)t = t_{0}\) and
\[
(1-r)^{n-1} * (1-r)^{n-1} * \ldots * (1-r)^{n-1} * r_{1} > 1 - s, 0 < s < 1
\]
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We have

\[ F_n(z_1, z_2, \ldots, z_n, t_0) > 1 - s \]

for all \( z_1, z_2, \ldots, z_n \in A \). Hence \( A \) is \( F \)-bounded.

**Definition 3.8.** Let \((X, F, *)\) be Generalized fuzzy \( k \)-metric space. Let
\[
< ((u_1)_n, (u_2)_n \ldots (u_k)_n, t_n) > \]

be a sequence in \( X^k \times (0, \infty) \) converging to a point
\((x_1, x_2, \ldots, x_k, t) \) in \( X^k \times (0, \infty) \). Then \( F_k \) is said to be continuous on \( X^k \times (0, \infty) \) if

\[
\lim_{n \to \infty} F_k((u_1)_n, (u_2)_n \ldots (u_k)_n, t_n) = F_k(x_1, x_2, \ldots, x_k, t)
\]

**Proposition 3.9.** Let \((X, F, *)\) be Generalized fuzzy \( k \)-metric space. Then \( F_k \) is continuous on \( X^k \times (0, \infty) \).

**Proof.** Let \( x_1, x_2, \ldots, x_k \in X \) and \( t > 0 \). Consider a sequence \(< ((v_1)_n, (v_2)_n \ldots (v_k)_n, s_n) > \)
in \( X^k \times (0, \infty) \) converging to \((x_1, x_2, \ldots, x_k, t) \). Then \(< F_k((v_1)_n, (v_2)_n \ldots (v_k)_n, s_n) > \)
is a sequence in \((0, 1] \). Hence there exists a subsequence \(< ((u_1)_n, (u_2)_n \ldots (u_k)_n, t_n) > \)
of the sequence \(< ((v_1)_n, (v_2)_n \ldots (v_k)_n, s_n) > \) such that \(< F_k((u_1)_n, (u_2)_n \ldots (u_k)_n, t_n) > \)
converges to some point in \([0, 1] \).

Now \( t_n \to t \) as \( n \to \infty \), hence for given \( \delta > 0 \) there exists \( n_0 \in \mathbb{N} \) such that
\[ |t_n - t| < \delta \]
for all \( n \geq n_0 \).

Let us choose \( \delta < \frac{t}{2} \). Then for all \( n \geq n_0 \), we have

\[
F_k((u_1)_n, (u_2)_n \ldots (u_k)_n, t_n) \geq F_k((u_1)_n, (u_2)_n \ldots (u_k)_n, t - \frac{(k + 1)\delta}{k})
\]

\[ \vdots \]

\[ \geq F_k((u_1)_n, x_1 \ldots x_1, \frac{\delta}{k}) * F_k((u_2)_n, x_2 \ldots x_2, \frac{\delta}{k}) * \ldots \]

\[ \ldots F_k((u_k)_n, x_k \ldots x_k, \frac{\delta}{k}) * F_k((x_1, x_2 \ldots x_k, t - 2\delta) \]

and

\[ F_k(x_1, x_2, \ldots x_k, t + 2\delta) \geq F_k(x_1, x_2, \ldots x_k, t_n + \delta) \]

\[ \geq F_k(x_1, (u_1)_n \ldots (u_1)_n, \frac{\delta}{k}) * F_k(x_2, (u_2)_n \ldots (u_2)_n, \frac{\delta}{k}) * \ldots \]

\[ \ldots * F_k(x_k, (u_k)_n \ldots (u_k)_n, \frac{\delta}{k}) * F_k((u_1)_n, (u_2)_n \ldots (u_k)_n, t_n) \]

Making \( n \to \infty \) we obtain

\[
\lim_{n \to \infty} F_k((u_1)_n, (u_2)_n \ldots (u_k)_n, t_n) = F_k((x_1, x_2 \ldots x_k, t - 2\delta) = F_k((x_1, x_2 \ldots x_k, t - 2\delta) \]

and

\[ F_k(x_1, x_2, \ldots x_k, t + 2\delta) \geq \lim_{n \to \infty} F_k((u_1)_n, (u_2)_n \ldots (u_k)_n, t_n) \]

respectively. Therefore

\[
F_k((x_1, x_2 \ldots x_k, t - 2\delta) \leq \lim_{n \to \infty} F_k((u_1)_n, (u_2)_n \ldots (u_k)_n, t_n) \leq F_k(x_1, x_2, \ldots x_k, t + 2\delta) \]

Since \( F_k(x_1, x_2, \ldots x_k, .): (0, \infty) \to [0, 1] \) is continuous, hence we have

\[
\lim_{n \to \infty} F_k((u_1)_n, (u_2)_n \ldots (u_k)_n, t_n) = F_k(x_1, x_2, \ldots x_k, t) \]

Therefore \( F_k \) is continuous on \( X^k \times (0, \infty) \). 

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Now we extend the Fuzzy Banach contraction theorem in the framework of Generalized fuzzy \( n \)-metric space.

**Theorem 3.10.** Let \((X, F_r, \ast)\) be a \(F_r\)-complete Generalized fuzzy \( r \)-metric space. Let \( T : X \to X \) be a mapping satisfying

\[
F_r(Tx, Tx, \ldots, Tx, kt) \geq F_r(x, x, \ldots, x, t)
\]  

for all \( x, x, \ldots, x \in X \) and \( 0 < k < 1 \). Then \( T \) has a unique fixed point.

**Proof.** Let \( y_0 \in X \). Consider a sequence \( \langle y_n \rangle \) in \( X \) such that \( y_n = T^n y_0 \). By Condition (3.5), we have

\[
F_r(Ty_{n-1}, Ty_{n}, \ldots, Ty_{n}, kt) \geq F_r(y_{n-1}, y_{n}, \ldots, y_{n}, t)
\]

\[
\implies F_r(y_{n}, y_{n+1}, \ldots, y_{n+1}, kt) \geq F_r(y_{n-1}, y_{n}, \ldots, y_{n}, t)
\]

\[
\implies F_r(y_{n}, y_{n+1}, \ldots, y_{n+1}, t) \geq F_r(y_{n-1}, y_{n}, \ldots, y_{n}, \frac{t}{k})
\]

\[
\quad \geq F_r(y_{n-2}, y_{n-1}, \ldots, y_{n-1}, \frac{t}{k^2})
\]

\[
\quad \vdots
\]

\[
\quad \geq F_r(y_0, y_1, \ldots, y_1, \frac{t}{k^n})
\]

We claim that the sequence \( \langle y_n \rangle \) is a \( F_r \)-Cauchy sequence \([12]\) in \( X \). For all natural numbers \( n \) and \( p \) we have

\[
F_r(y_{n}, y_{n+p}, \ldots, y_{n+p}, t) \geq F_r(y_{n}, y_{n+1}, \ldots, y_{n+1}, \frac{t}{p-1}) * F_r(y_{n+1}, y_{n+2}, \ldots, y_{n+p}, \frac{t(p-2)}{p-1})
\]

\[
\quad \geq F_r(y_{n}, y_{n+1}, \ldots, y_{n+1}, \frac{t}{p-1}) * F_r(y_{n+1}, y_{n+2}, \ldots, y_{n+2}, \frac{t}{p-1})
\]

\[
\quad \ast F_r(y_{n+2}, y_{n+p}, \ldots, y_{n+p}, \frac{t(p-2)}{p-1})
\]

\[
\quad \vdots
\]

\[
\quad \geq F_r(y_{n}, y_{n+1}, \ldots, y_{n+1}, \frac{t}{p-1}) * F_r(y_{n+1}, y_{n+2}, \ldots, y_{n+2}, \frac{t}{p-1})
\]

\[
\quad \ast \ldots * F_r(y_{n+p-1}, y_{n+p}, \ldots, y_{n+p}, \frac{t}{p-1})
\]

\[
\quad \geq F_r(y_0, y_1, \ldots, y_1, \frac{t}{k^n(p-1)}) * F_r(y_0, y_1, \ldots, y_1, \frac{t}{k^{n+1}(p-1)})
\]

\[
\quad \ast \ldots * F_r(y_0, y_1, \ldots, y_1, \frac{t}{k^{n+p-1}(p-1)})
\]

Since the \( t \)-norm \( \ast \) is continuous and \( F_r(x, x, \ldots, x) \) is continuous, Hence making \( n \to \infty \), we have

\[
\lim_{n \to \infty} F_r(y_{n}, y_{n+p}, \ldots, y_{n+p}, t) \geq 1 \ast 1 \ast \ldots 1 = 1
\]

This shows that \( \langle y_n \rangle \) is a \( F_r \)-Cauchy sequence. Since \( X \) is \( F_r \)-Complete, there exists a point \( u \in X \) such that \( \langle y_n \rangle \) is \( F_r \)-convergent and converges to \( u \) \([12]\).
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suppose \( Tu \neq u \) then
\[
F_r(Tu,u,...,u,t) \geq F_r(Tu,Ty_1,...,Ty_n,Tu,t) \geq F_r(u,y_1,...,y_n,Tu,Tu,t) \geq F_r(u,y_1,...,y_n,Tu,Tu,t) \geq F_r(u,y_1,...,y_n,Tu,Tu,t) \geq ...
\]
Taking the limit as \( n \to \infty \) and using the fact that function \( F_r \) is continuous on its variables, we have

\[ F_r(Tu,u,...,u,t) \to 1 \]
This implies \( Tu = u \).

Next to show the uniqueness, suppose there exists \( v \in X \) such that \( Tv = v \). Then
\[
F_r(v,u,...,u,t) = F_r(Tv,Tu,...,Tu,t)
\]
\[
\geq F_r(v,u,...,u,t/k)
\]
\[
= F_r(Tv,Tu,...,Tu,t/k)
\]
\[
\geq F_r(v,u,...,u,t/k^2)
\]
\[
\geq ...
\]
\[
\geq F_r(v,u,...,u,t/k^n) \to 1 \quad \text{as} \quad n \to \infty
\]
Which implies \( v = u \). \( \square \)

We now give an example to illustrate Theorem 3.14.

**Example 3.11.** Let \( X = [-1, 1] \), \( a * b = ab \) for all \( a,b \in [0,1] \) and \( t > 0 \), define
\[
F_n(x_1,x_2,...,x_n,t) = \frac{t}{t + \sum_{i,j=1,i<j}^{n} |x_i - x_j|}
\]
for all \( x_1,x_2,...,x_n \in X \). Clearly \( (X,F_n,*) \) is a \( F_n \)-complete Generalized fuzzy \( n \)-metric space. Define a mapping \( T : X \to X \) by \( Tx = x/6 \) for all \( x \in X \). One can see that the condition 3.5 holds for all \( x_1,x_2,...,x_n \in X \) and \( \frac{1}{6} \leq k < 1 \) and 0 is the unique fixed point of \( T \).

4. **APPLICATION TO VECTOR IMAGE FILTERING**

The quality of digital images is affected by the sensor noise and the channel noise. The sensor noise is produced during image formation while the channel noise is produced during transmission. Therefore, it is essential to reduce the noise for estimating the original image information from noisy data. This process is called the colour image filtering and is an important part of any colour image processing system ([17],[20]).

There are several approaches to construct a colour filter for this purpose. The vector approach ([2],[14],[19]) is observed to be more appropriate compared to other traditional approaches. The most well known filter following this approach is a Vector Median Filter (VMF). In this approach the colour images are treated as a vector field and a window is moved over the input image. The vector filter selects the output vector on the basis of ordering of vectors in the defined sliding window.

Let \( W \) be the sliding window of size \( n \) and let \( x_i, i = 1,2,...,n \) be the pixels in \( W \). Let the vector valued image function at pixel \( x_j \) be denoted by \( I_j = \ldots \)
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\((I_j(1), I_j(2), I_j(3))\). For RGB images, \(I_j \in \{0, 1, \ldots, 255\}^3\). The nearness or closeness degree between two vectors \(I_i\) and \(I_j\) is described by

\[
d(I_i, I_j) = \|I_i - I_j\| = \left(\sum_{l=1}^{3} |I_i(l) - I_j(l)|^p\right)^{1/p}
\]

Let the aggregated distance associated with the input vector \(I_i\) is given by

\[
D^i = \sum_{j=1}^{n} d(I_i, I_j)
\]

Then the order sequence of aggregated distances \(D^{(1)} \leq D^{(2)} \leq \cdots \leq D^{(n)}\) implies the same ordering of corresponding vectors \(I^{(1)} \leq I^{(2)} \leq \cdots \leq I^{(n)}\). The VMF outputs the vector \(\hat{I}_k\) that minimizes the aggregated distances to the other vectors in \(W\), i.e. \(I_{out} = \hat{I}_k\) for which

\[
k = \arg\min_{i} D^i, \quad i = 1, 2, \ldots, n
\]

Hence the lowest ranked vector \(I^{(1)}\) is the output of VMF.

It has been observed that fuzzy based methods ([11], [3], [12], [16], [21], [28], [29]) are useful in detection and removal of noise during image processing. Morillas et al ([10]) replaced the classical metric defining nearness between pixels \(I_i\) and \(I_j\) by the following fuzzy metric-

\[
M(I_i, I_j) = \prod_{l=1}^{3} \frac{\min\{I_i(l), I_j(l)\} + K}{\max\{I_i(l), I_j(l)\} + K}
\]

(4.1)

Where \(I_i(l), I_j(l) \in \{0, 255\}\) for the processing of RGB images and \(K > 0\). Recently Khan [12] proposed to replace the fuzzy metric \(M(I_i, I_j)\) by generalized fuzzy metric \(F^{(3)}(I_1, I_2, \ldots, I_r)\) defined in proposition 2.9 Then the accumulated fuzzy measure associated to the vector \(I_i\) is given by

\[
D^i = \sum_{\substack{1 \leq i_1 < i_2 < \ldots < i_{r-1} \leq n \\ i_k \neq i, \ k = 1, 2, \ldots, r-1}} F^{(3)}(I_i, I_{i_1}, \ldots, I_{i_{r-1}})
\]

(4.2)

For different values of \(r\), We have different algorithms reducing the noise in colour image processing.

Now we propose an algorithm by choosing \(r = 4\) for a 8-neighbourhood \(3 \times 3\) window. For this choice of \(r(= 4)\), we have a new vector filter with accumulated fuzzy measure associated to the vector \(I_i\) as

\[
D^i = \sum_{\substack{1 \leq i_1 < i_2 < i_3 \leq 9 \\ i_k \neq i, \ k = 1, 2, 3}} F^{(3)}(I_i, I_{i_1}, I_{i_2}, I_{i_3})
\]

(4.3)

Where \(I_{ip} = (I_{ip}(1), I_{ip}(2), I_{ip}(3))\), \(I_{ip}(l) \in \{0, 1, 2, \ldots, 255\}\) for \(l = 1, 2, 3; \ p = 1, 2, 3\), and

\[
F^{(3)}(I_i, I_{i_1}, I_{i_2}, I_{i_3}) = \prod_{l=1}^{3} \frac{\min\{I_i(l), I_{i_1}(l), I_{i_2}(l), I_{i_3}(l)\} + K}{\max\{I_i(l), I_{i_1}(l), I_{i_2}(l), I_{i_3}(l)\} + K}
\]

(4.4)

for the processing of RGB images and \(K > 0\). The appropriate value of \(K\) can be decided by analyzing the performance (MSE) of the fuzzy metric \(F^{(3)}\) with respect to different values of \(K\). The fuzzy metric \(F^{(3)}\) is a particular stationary form of the
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generalized metric \( F_4(x, y, z, w, t) \). An interpretation of such metric can be given as follows-

\[
F_4(x, y, z, w, t) = \alpha \text{ if and only if the probability } P(\text{Perimeter of the rectangle with vertices } x, y, z \text{ and } w) \leq t = \alpha.
\]

Then the accumulated fuzzy measure \( D_i \) associated with the vector \( I_i \) will represent the sum of fuzzy analogue perimeters of all the rectangles formed with the pixel \( I_i \) as one vertex.

Let us consider a 8-neighbourhood \( 3 \times 3 \) window with pixels \( \{(i,j) : i,j \in \{1,2,3\}\} \). If we number the pixel \((i,j)\) as \((3i+j-3)\), then the pixels numbered \(1,3,7\) and \(9\) will have similar spatial neighbourhoods. The pixels numbered \(2,4,6\) and \(8\) are also spatially similar with respect to the sliding window. The pixel numbered \(5\) is the central pixel. Now we propose the algorithm for computing \( D_i \) as follows-

\[
D^1 = F_4(3)(I_1, I_2, I_4, I_5) + F_4(3)(I_1, I_3, I_7, I_9) + F_4(3)(I_1, I_1, I_6, I_8),
\]

\[
D^2 = F_4(3)(I_2, I_1, I_3, I_5) + F_4(3)(I_2, I_4, I_6, I_8) + F_4(3)(I_2, I_2, I_7, I_9)
\]

\[
D^5 = F_4(3)(I_5, I_1, I_2, I_4) + F_4(3)(I_5, I_6, I_8, I_9) + F_4(3)(I_5, I_3, I_5, I_7)
\]

We can follow a similar approach to compute \( D_i \) for other spatially similar pixels numbered \(3,7,9\) and \(4,6,8\).

The filter output will be \( \tilde{I}_k \in W \) that maximizes the aggregated fuzzy measure to other vectors in \( W \), i.e. \( I_{out} = \tilde{I}_k \) for which

\[
k = \arg \max_i (D^i), \quad i = 1, 2, ..., 9
\]

The ordering \( D^1 : D^{(1)} \geq D^{(2)} \geq \cdots \geq D^{(9)} \) with fuzzy rule based order statistics (18, 20) implies the ordering \( I_i : I_{(1)} \geq I_{(2)} \geq \cdots \geq I_{(9)} \). Hence in light of relation \((4.5)\), the vector \( I_{(1)} \) is the output vector.

The proposed filter can be analyzed for computational complexity and performance analysis by using standard measures like Mean Absolute Error(MAE), Peak Signal to Noise Ratio(PSNR) and Normalized Colour Difference(NCD).

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