ON gg-CONTINUOUS AND gg-IRRESOLUTE MAPS IN TOPOLOGICAL SPACES

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Abstract: In this research paper, a new class of continuous functions called gg-continuous maps in topological space are introduced and studied. Also some of their properties have been investigated. We also introduce gg-irresolute maps, strongly gg-continuous maps, perfectly gg-continuous maps and discuss some properties.

Keywords: gg-closed sets, gg-open sets, gg-continuous maps, gg-irresolute maps, strongly gg-continuous maps and perfectly gg-continuous maps.

1. INTRODUCTION

The concept of continuous functions play a very important role in general topology. The regular continuous and completely continuous functions are introduced and studied by Arya S P [2]. Later, R S walli et al all [31] introduced and investigated rw-continuous functions in topological space. Recently, Basavaraj M Ittanagi et al [5] introduced and studied the basic properties of gg-closed sets in topological space. The aim of this paper is to introduce and study basic properties of gg-continuous and irresolute maps in topological space.

2. PRLIMINARIES

In this paper X or (X, τ) and Y or (Y, σ) denote topological spaces on which no separation axioms are assumed. For a subset A of a topological space X, cl(A), int(A), X-A or A c represent closure of A, interior of A and complement of A in X respectively.

Definition 2.1: A subset A of a topological space (X, τ) is called a
1. Regular open set [24] if A=int(cl(A)) and regular closed if A=cl(int(A))
2. Regular semi open set [8] if there exists a regular open set U such that U A cl(U)
3. Generalized closed set (g-closed) [16] if cl(A) U whenever A U and U is open in (X, τ).
4. gg-closed set [5] if gcl(A) U whenever A U and U is regular semi open in (X, τ).

The complement of the closed sets mentioned above are their open sets respectively and vice versa.

Definition 2.2 A function f: (X, τ)→(Y, σ) is called a
1. Continuous [15] if f(V) is closed in X for every closed subset V of Y.
2. Regular continuous [2] if f(V) is r-closed in X for every closed subset V of Y.
3. Completely continuous [2] if f(V) is regular closed in X for every closed subset V of Y.
4. α continuous [14] if f(V) is α-closed in X for every closed subset V of Y.
5. Semi continuous [15] if f(V) is semi closed in X for every closed subset V of Y.
6. Semi pre continuous [1] if f(V) is semi pre closed in X for every closed subset V of Y.
7. Strongly Continuous [24] if f(V) is clopen in X for every subset V of Y.
8. g-continuous [4] if f(V) is g closed in X for every closed subset V of Y.
9. w-continuous [28] if f(V) is w closed in X for every closed subset V of Y.
10. gr-continuous [22] if f(V) is gr closed in X for every closed subset V of Y.
11. g*-continuous [30] if f(V) is g* closed in X for every closed subset V of Y.
12. swg*-continuous [19] if f(V) is swg* closed in X for every closed subset V of Y.
13. βwg*-continuous [11] if f(V) is βwg* closed in X for every closed subset V of Y.
14. rαg-continuous [21] if f(V) is rαg closed in X for every closed subset V of Y.
15. rwg-continuous [20] if f(V) is rwg closed in X for every closed subset V of Y.
Theorem 3.5

Not a continuous function as the closed set \( F = \{ q \} \) in \( Y \), \( f : X \rightarrow Y \) defined by \( f(p) = q \), \( f(q) = r \), \( f(r) = r \) is \( gg \) continuous but not conversely.

Example 3.4

Let \( X = Y = \{ p, q, r \} \). Let \( \tau = \{ \varnothing, X, \{ p \}, \{ q \}, \{ p, q \}, \{ p, r \} \} \) be a topology on \( X \) and \( \sigma = \{ \varnothing, Y, \{ p \}, \{ q \}, \{ p, q \}, \{ p, r \} \} \) be a topology on \( Y \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function defined by \( f(p) = q \), \( f(q) = q \), \( f(r) = r \) is \( gg \) continuous but not a continuous function as the closed set \( F = \{ q \} \) in \( Y \), \( f^{-1}(F) = \{ p, q \} \) is not a closed set in \( X \).

Definition 3.2

A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called a

1. Irresolute if \( f^{-1}(V) \) is semi closed in \( X \) for every semi closed subset \( V \) of \( Y \).
2. \( w \)-Irresolute [28] if \( f^{-1}(V) \) is \( w \)-closed in \( X \) for every \( w \)-closed subset \( V \) of \( Y \).
3. \( gc \)-Irresolute [27] if \( f^{-1}(V) \) is \( g \)-closed in \( X \) for every \( g \)-closed subset \( V \) of \( Y \).
4. Contra \( w \) Irresolute [28] if \( f^{-1}(V) \) is \( w \)-open in \( X \) for every \( w \)-closed subset \( V \) of \( Y \).
5. Contra Irresolute [14] if \( f^{-1}(V) \) is semi open in \( X \) for every semi closed subset \( V \) of \( Y \).
6. Contra \( r \)-Irresolute [2] if \( f^{-1}(V) \) is regular open in \( X \) for every regular closed subset \( V \) of \( Y \).
7. Contra continuous [13] if \( f^{-1}(V) \) is open in \( X \) for every closed subset \( V \) of \( Y \).

Results 2.4 [5]

1) Every closed (respectively regular closed, \( g \)-closed, \( w \)-closed, \( gr \)-closed, \( g^* \)-closed, \( swg^* \)-closed and \( \beta \) \( swg^* \) closed) set is \( gg \)-closed set in \( X \).
2) Every \( gg \)-closed set is \( r^*g \)-closed (respectively \( rwg \)-closed and \( \beta \) \( wg^* \) closed) set in \( X \).

Results 2.5 [5]

Let \( A \) be a subset of a topological space \((X, \tau)\)

1) If \( A \) is semi open and \( swg^* \)-closed then \( A \) is \( gg \)-closed in \( X \).
2) If \( A \) is regular open and \( gg \)-closed then \( A \) is \( g \)-closed in \( X \).
3) If \( A \) is open and \( g \)-closed then \( A \) is \( gg \)-closed in \( X \).
4) If \( A \) is open and \( gr \)-closed then \( A \) is \( gg \)-closed in \( X \).
5) If \( A \) is regular open and \( gg \)-closed then \( A \) is \( re \)gular closed in \( X \).
6) If \( A \) is semi open and \( w \)-closed then \( A \) is \( gg \)-closed in \( X \).
7) If \( A \) is semi open and \( swg^* \)-closed then \( A \) is \( gg \)-closed in \( X \).

3. \( gg \)-CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

Definition 3.1

A function \( f \) from a topological space \( X \) in to a topological space \( Y \) is called a \( gg \)-continuous if inverse image of every closed subset in \( Y \) is a \( gg \)-closed set in \( X \).

Example 3.2

Let \( X = Y = \{ p, q, r \} \). Let \( \tau = \{ \varnothing, X, \{ p \}, \{ q \}, \{ p, q \}, \{ p, r \} \} \) be a topology on \( X \) and \( \sigma = \{ \varnothing, Y, \{ p \}, \{ q \}, \{ p, q \} \} \) be a topology on \( Y \). Let \( f : X \rightarrow Y \) defined by \( f(p) = q \), \( f(q) = r \), \( f(r) = r \) is \( gg \)-continuous.

Theorem 3.3

Every continuous function is \( gg \)-continuous but not conversely.

Proof: Let \( f : X \rightarrow Y \) be continuous and \( F \) be any closed set in \( Y \). Then \( f^{-1}(F) \) is closed set in \( X \). Since every closed set in \( X \) is \( gg \)-closed then \( f^{-1}(F) \) is \( gg \)-closed set in \( X \). Therefore \( f \) is \( gg \)-continuous.

Example 3.4

Let \( X = Y = \{ p, q, r \} \). Let \( \tau = \{ \varnothing, X, \{ p \}, \{ q \}, \{ p, q \}, \{ p, r \} \} \) be a topology on \( X \) and \( \sigma = \{ \varnothing, Y, \{ p \}, \{ q \}, \{ p, q \}, \{ p, r \} \} \) be a topology on \( Y \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function defined by \( f(p) = q \), \( f(q) = q \), \( f(r) = r \) is \( gg \)-continuous but not a continuous function as the closed set \( F = \{ q \} \) in \( Y \), \( f^{-1}(F) = \{ p, q \} \) is not a closed set in \( X \).

Theorem 3.5

Every \( g \)-continuous is \( gg \)-continuous but not conversely.
Proof: The proof follows from the fact that every g-closed set is gg-closed set.

**Example 3.6** In example 3.4, f is gg-continuous but not a g-continuous as the closed set F={q} in Y, \( f^{-1}(F) = \{p, q\} \) is not a g-closed set in X.

**Theorem 3.7** Every w-continuous is gg-continuous but not conversely.

Proof: The proof follows from the fact that every w-closed set is gg-closed set.

**Example 3.8** In example 3.4, f is gg-continuous but not a w-continuous as the closed set F={q} in Y, \( f^{-1}(F) = \{p, q\} \) is not a w-closed set in X.

**Theorem 3.9** Every gr-continuous is gg-continuous but not conversely.

Proof: The proof follows from the fact that every gr-closed set is gg-closed set.

**Example 3.10** In example 3.4, f is gg-continuous but not a gr-continuous as the closed set F={q} in Y, \( f^{-1}(F) = \{p, q\} \) is not a gr-closed set in X.

**Theorem 3.11** Every g*-continuous is gg-continuous but not conversely.

Proof: The proof follows from the fact that every g*-closed set is gg-closed set.

**Example 3.12** In example 3.4, f is gg-continuous but not a g*-continuous as the closed set F={q} in Y, \( f^{-1}(F) = \{p, q\} \) is not a g*-closed set in X.

**Theorem 3.13** Every swg*-continuous is gg-continuous but not conversely.

Proof: The proof follows from the fact that every swg*-closed set is gg-closed set.

**Example 3.14** In example 3.4, f is gg-continuous but not a swg*-continuous as the closed set F={q} in Y, \( f^{-1}(F) = \{p, q\} \) is not a swg*-closed set in X.

**Theorem 3.15** Every \( \beta \)wg*-continuous is gg-continuous but not conversely.

Proof: The proof follows from the fact that every \( \beta \)wg*-closed set is gg-closed set.

**Example 3.16** In example 3.4, f is gg-continuous but not a \( \beta \)wg*-continuous as the closed set F={q} in Y, \( f^{-1}(F) = \{p, q\} \) is not a \( \beta \)wg*-closed set in X.

**Theorem 3.17** Every regular-continuous is gg-continuous but not conversely.

Proof: The proof follows from the fact that every r-closed set is gg-closed set.

**Example 3.18** In example 3.4, f is gg-continuous but not a r-continuous as the closed set F={q} in Y, \( f^{-1}(F) = \{p, q\} \) is not a r-closed set in X.

**Theorem 3.19** If a map \( f: X \to Y \) is continuous then the following holds.

i) If \( f \) is gg-continuous then it is r^g-continuous but not conversely.

ii) If \( f \) is gg-continuous then it is rwg-continuous but not conversely.

iii) If \( f \) is gg-continuous then it is \( \beta \)wg**-continuous but not conversely.

Proof:

i) Let \( F \) be a closed set in \( Y \). Since \( f \) is gg-continuous then \( f^{-1}(F) \) is gg-closed in \( X \). Since every gg-closed set is r^g closed, then \( f^{-1}(F) \) is r^g closed in \( X \). Hence \( f \) is r^g continuous. Similarly we can prove ii) and iii).
Example 3.20 Let X=Y={p, q, r}. Let τ={ϕ, X, {p}, {q}, {p, q}, {p, r}} be a topology on X and σ={ϕ, Y, {p}, {q}, {p, q}} be a topology on Y. Let f: (X, τ)→(Y, σ) be a function defined by f(p)=r, f(q)=p, f(r)=q is r^g-continuous, rwg-continuous and βwg** continuous but not a gg-continuous function as the closed set F={r} in Y, f^(-1)(F)={p} is not a gg-closed set in X.

Remark 3.21 The following Examples show that gg-continuous is independent with some existing continuous functions in topological spaces.

Example 3.22 Let X=Y={p, q, r}. Let τ={ϕ, X, {p}, {q}, {p, q}} be a topology on X and σ={ϕ, Y, {p}, {q}, {p, q}, {a, c}} be a topology on Y. Let f: (X, τ)→(Y, σ) be a function defined by f(p)=q, f(q)=r, f(r)=p is gg-continuous but not a semi continuous, semi pre continuous, α-continuous, sg-continuous, gs-continuous, α g-continuous, gα-continuous, wα-continuous, gsp-continuous, gp-continuous, swg-continuous, wg-continuous, g*p-continuous, rps-continuous, αrw-continuous, ρ-continuous as the closed set F={b, c} in Y, f^(-1)(F)={a, b} is not a semi closed (respectively semi pre closed, α-closed, sg-closed, gs-closed, αg-closed, gα-closed, wα-closed, gsp-closed, gp-closed, swg-closed, wg-closed, g*p-closed, rps-closed, αrw-closed, ρ-closed) set in X.

Example 3.23 Let X={a, b, c, d} and τ={ϕ, X, {a}, {b}, {a, b}, {a, b, c}} be a topology on X. Let σ={ϕ, Y, {a, b}} be a topology on Y. Let f: (X, τ)→(Y, σ) be a function defined by f(a)=a, f(b)=a, f(c)=c, f(d)=b is semi continuous, semi pre continuous, α-continuous, sg-continuous, gs-continuous, αg-continuous, gα-continuous, wα-continuous, gsp-continuous, gp-continuous, swg-continuous, wg-continuous, g*p-continuous, rps-continuous, αrw-continuous, ρ-continuous but not a gg-continuous as the closed set F={c} in Y, f^(-1)(F)={c} is not a gg-closed set in X.

Remark 3.24 From the above discussions and known facts, the relation between gg-continuous and some existing continuous functions in topological space is shown in the following figure.

Theorem 3.25 Let f: X→Y be a map. Then the following statements are equivalent.
i) f is gg-continuous 

ii) The inverse image of each open set in Y is gg-open in X.

Proof:

i) Let f: X→Y be a gg-continuous. Let U be an open set in Y, then U^c is closed in Y. Since f is gg-continuous, f^(-1)(U^c) is gg-closed in X. But f^(-1)(U^c)=X-f^(-1)(U). Thus f^(-1)(U) is gg-open in X. 

ii) Suppose that inverse image of each open set in Y is gg-open in X. Let V be any closed set in Y. By assumption f^(-1)(V) is gg-open in X. But f^(-1)(V)=X-f^(-1)(V). Thus X-f^(-1)(V) is gg-open in X and so f^(-1)(V) is gg-closed in X. Thus f is gg-continuous. Hence the Proof.

Theorem 3.26 If f: (X, τ)→(Y, σ) is a map then the following holds.

i) If f is gg-continuous and contra r-irresolute map then f is g-continuous.

ii) If f is g-continuous and contra continuous map then f is gg-continuous.

iii) If f is swg* continuous and contra irresolute map then f is gg-continuous.

iv) If f is gg-continuous and contra r-irresolute map then f is regular continuous.

v) If f is gr-continuous and contra continuous then f is gg-continuous.

vi) If f is w-continuous and contra irresolute then f is gg-continuous.

vii) If f is w-irresolute then it is gg-continuous.

Proof: Let V be any regular closed set of Y. Since every regular closed set is closed, V is closed set in Y. Also f(A) is closed set in X. By results 2.5 [5] f^(-1)(V) is g-closed in X. Thus f is g-continuous.

Theorem 3.27 If f: (X, τ)→(Y, σ) be a map. Then the following statements are equivalent

i) For each point x∈X and each open set V in Y with f(x)∈V, there is a gg-open set U containing x such that x∈U and f(U)⊆V.

ii) For each subset A of X, f(ggcl(A))⊆cl(f(A))

iii) For each subset B of Y, ggcl(f^(-1)(B)) ⊆ f^(-1)(cl(B))

Proof:

(i)→(ii) Suppose (i) holds and let y∈f(ggcl(A)) and V be an open set containing y. From (i), there exists x∈ggcl(A) such that f(x)=y and gg-open set U containing x such that x∈U and f(U)⊆V and x∈ggcl(A). Then by theorem 3.29, U∩A≠ φ. That is φ ≠ f(U∩A)⊆f(U)∩f(A)⊆V∩f(A). Therefore f(ggcl(A))⊆cl(f(A)).

(ii)→(i) Suppose (ii) holds and V be an open set in Y containing f(x). Let A∈f^(-1)(V). This implies that x∉A. Since f(ggcl(A))⊆cl(f(A)) ⊆ V. This implies that ggcl(A)⊆f^(-1)(V)=A. Since x∉A implies that x∉ggcl(A) and by theorem 3.29, there exists a gg-open set U containing x such that U∩A≠ φ then U⊆A^c and hence f(U)⊆f(A^c)⊆V.
(ii) $\implies$(iii)
Suppose (ii) holds. Let $B$ be any subset of $Y$. Replacing $A$ by $f^{-1}(B)$ in (ii) we get $f(ggcl(f^{-1}(B))) \subseteq \text{cl}(f(f^{-1}(B))) \subseteq \text{cl}(B)$. Hence $ggcl(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$.

(iii) $\implies$(ii)
Suppose (iii) holds. Let $B=f(A)$ where $A$ is a subset of $X$. Then from (iii) we get $ggcl(f^{-1}(f(A))) \subseteq f^{-1}(\text{cl}(f(A)))$. That is $ggcl(A) \subseteq f^{-1}(cl(f(A)))$. Therefore $ggcl(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$.

**Definition 3.31** Let $(X, \tau)$ be a topological space and $\tau_{gg} = \{ V \subseteq X / ggcl(V) = V^c \}$ is a topology on $X$.

**Definition 3.32** A topological space $(X, \tau)$ is called a $gg_{Tc}$ space if every $gg$-closed is closed.

**Definition 3.33** A topological space $(X, \tau)$ is called a $gg_{Tcg}$ space if every $gg$-closed is $g$-closed in $X$.

**Remark 3.34** The composition of two $gg$-continuous maps need not be continuous.

**Example 3.35** Let $X = Y = Z = \{ p, q, r \}$. Let $\tau = \{ \emptyset, X, \{ p \}, \{ q \}, \{ p, q \}, \{ p, r \} \}$ be a topology on $X$, $\sigma = \{ \emptyset, Y, \{ q \}, \{ p, q \}, \{ p, r \} \}$ be a topology on $Y$, and $\eta = \{ \emptyset, Z, \{ p \}, \{ q \}, \{ p, q \}, \{ p, r \} \}$ be a topology on $Z$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$, $g: (Y, \sigma) \rightarrow (Z, \eta)$ and $gof: (X, \tau) \rightarrow (Z, \eta)$ are identity functions. Both $f$ and $g$ are $gg$-continuous but $gof$ is not a $gg$-continuous map as the closed set $F = \{ q \}$ in $Z$, $(gof)^{-1}(F) = \{ q \}$ is not $gg$-closed set in $X$.

**Theorem 3.36** Let $f: X \rightarrow Y$ be $gg$-continuous and $g: Y \rightarrow Z$ is continuous then $gof: X \rightarrow Z$ is $gg$-continuous.

Proof: Let $V$ be any open set in $Z$. Since $g$ is continuous, $g^{-1}(V)$ is open in $Y$. Since $f$ is $gg$-continuous, $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is $gg$-open in $X$. Hence $gof$ is $gg$-continuous.

**Theorem 3.37** Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be $gg$-continuous functions and $Y$ be $gg_{Tc}$ space then $gof: X \rightarrow Z$ is $gg$-continuous.

Proof: Let $V$ be any open set in $Z$. Since $g$ is $gg$-continuous, $g^{-1}(V)$ is $gg$-open in $Y$ and $Y$ is $gg_{Tc}$ space, then $g^{-1}(V)$ is open in $Y$. Since $f$ is $gg$-continuous $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is $gg$-open in $X$. Hence $gof$ is $gg$-continuous.

**Definition 3.38** A function $f: X \rightarrow Y$ is called a perfectly $gg$-continuous if $f^{-1}(V)$ is clopen (open and closed) set in $X$ for every $gg$-open set $V$ in $Y$.

**Theorem 3.39** If $f: X \rightarrow Y$ is continuous then the following holds.

i) If $f$ is perfectly $gg$-continuous then it is $gg$-continuous.

ii) If $f$ is perfectly $gg$-continuous then it is $r^*g$-continuous (rwg-continuous and $\beta$ $wg^{**}$-continuous).

Proof: i) Let $U$ be open set in $Y$. Since $f$ is perfectly continuous then $f^{-1}(U)$ is both open and closed in $X$. Since every open is $gg$-open, $f^{-1}(U)$ is $gg$-open in $X$. Hence $f$ is $gg$-continuous.

Similarly we can prove ii).

**Definition 3.40** A function $f: X \rightarrow Y$ is called $gg^{*}$-continuous if $f^{-1}(V)$ is $gg$-closed set in $X$ for every $g$-closed set $V$ in $Y$.

**Theorem 3.41** If $f: X \rightarrow Y$ is $gg^{*}$-continuous then it is $gg$-continuous but converse is not true.

Proof: Let $f: X \rightarrow Y$ be $gg^{*}$-continuous. Let $F$ be any closed set in $Y$. Since $f$ is $gg^{*}$-continuous, $f^{-1}(F)$ is $gg$-closed set in $X$. Since every closed set is $g$-closed set in $Y$ then the inverse image $f^{-1}(F)$ is $gg$-closed set in $X$. Hence $f$ is $gg$-continuous.

**Example 3.42** Let $X = Y = \{ p, q, r \}$. Let $\tau = \{ \emptyset, X, \{ p \}, \{ q \}, \{ p, q \} \}$ be a topology on $X$ and $\sigma = \{ \emptyset, Y, \{ q \}, \{ p, q \}, \{ p, r \} \}$ be a topology on $Y$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by $f(p) = r$, $f(q) = q$, $f(r) = r$ is $gg$-continuous but not a $gg^{*}$-continuous function as the $g$-closed set $F = \{ q \}$ in $Y$, $f^{-1}(F) = \{ q \}$ is not a $gg$-closed set in $X$. 

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Definition 3.43: A function $f: X \rightarrow Y$ is called a gg-irresolute map if $f^1(V)$ is gg-closed in $X$ for every gg-closed set $V$ in $Y$.

Definition 3.44: A function $f: X \rightarrow Y$ is called a strongly gg-continuous map if $f^1(V)$ is closed set in $X$ for every gg-closed set $V$ in $Y$.

Theorem 3.45: If $f: (X, \tau) \rightarrow (Y, \sigma)$ is gg-irresolute then it is gg-continuous but not conversely.

Proof: Let $f: X \rightarrow Y$ be gg-irresolute. Let $F$ be any closed set in $Y$ and hence gg-closed in $Y$. Since $f$ is gg-irresolute, $f^{-1}(V)$ is gg-closed set in $X$. Therefore $f$ is gg-continuous.

Example 3.46: Let $X=\{p, q, r\}$. Let $\tau=\{\varnothing, X, \{p\}, \{q\}, \{p, q\}, \{p, r\}\}$ be a topology on $X$ and $\sigma=\{\varnothing, Y, \{p\}, \{q\}, \{p, q\}, \{p, r\}\}$ be a topology on $Y$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by $f(p)=p, f(q)=r, f(r)=r$ is gg-continuous but not a gg-irresolute map as the gg-closed set $F=\{q\}$ in $Y$, $f^{-1}(F)=\{p\}$ is not a gg-closed set in $X$.

Theorem 3.47: If $f: (X, \tau) \rightarrow (Y, \sigma)$ is gg-irresolute if and only if $f^{-1}(V)$ is gg-open set in $X$ for every open set $V$ in $Y$.

Proof: Suppose that $f: X \rightarrow Y$ is gg-irresolute and $U$ be gg-open set in $Y$. Then $U^c$ is gg-closed in $Y$. By the definition of gg-irresolute, $f^{-1}(U^c)$ is gg-closed in $X$. But $f^{-1}(U^c)=X-f^{-1}(U)$. Thus $f(U)$ is gg-open in $X$.

Conversely
Suppose that $f^{-1}(F)$ is gg-open in $X$ for every gg-open set $F$ in $Y$. Let $F$ be any gg-closed set in $Y$. By the definition, $f^{-1}(F)$ is gg-open in $X$. But $f^{-1}(F)=X-f^{-1}(F)$. Thus $X-f^{-1}(F)$ is gg-open in $X$ and hence $f^{-1}(F)$ is gg-closed in $X$. therefore $f$ is gg-irresolute.

Theorem 3.48: If $f: (X, \tau) \rightarrow (Y, \sigma)$ is gg-irresolute then it is gg*-continuous but not conversely.

Proof: Let $f: X \rightarrow Y$ be gg-irresolute. Let $F$ be any g-closed set in $Y$ and hence $f$ is gg-closed in $Y$. By the definition of gg-irresolute, $f^1(V)$ is gg-closed set in $X$. Therefore $f$ is gg*-continuous.

Example 3.49: Let $X=\{p, q, r\}$. Let $\tau=\{\varnothing, X, \{p\}, \{q\}, \{p, q\}\}$ and $\sigma=\{\varnothing, Y, \{p\}, \{q\}, \{p, q\}, \{p, r\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by $f(p)=p, f(q)=q, f(r)=r$ is gg*-continuous but not a gg-irresolute map as the gg-closed set $F=\{p, q\}$ in $Y$, $f^{-1}(F)=\{p\}$ is not a gg-closed set in $X$.

Theorem 3.50: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ is gg-irresolute then $f(ggcl(A)) \subseteq ggcl(f(A))$ for every subset $A$ of $X$.

Proof: Let $A \subseteq X$ and $ggcl(f(A))$ is gg-closed in $Y$. Since $f$ is gg-irresolute, $f^{-1}(ggcl(A))$ is gg-closed in $X$. Further $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(ggcl(f(A)))$. By the definition of gg-closure, $ggcl(A) \subseteq f^{-1}(ggcl(f(A)))$. Hence $f(ggcl(A)) \subseteq ggcl(f(A))$.

Theorem 3.51: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ and $g: (Y, \sigma) \rightarrow (Z, \eta)$ be any two functions. Then

1) $gof: (X, \tau) \rightarrow (Z, \eta)$ is gg-irresolute if $g$ is gg-irresolute and $f$ is gg-irresolute

2) $gof: (X, \tau) \rightarrow (Z, \eta)$ is gg-continuous if $g$ is gg-continuous and $f$ is gg-irresolute.

Proof: (i) Let $F$ be any gg-closed set in $(Z, \eta)$. Since $g$ is gg-irresolute then $g^{-1}(F)$ is gg-closed set in $(Y, \sigma)$. Since $f$ is gg-irresolute $f^{-1}(g^{-1}(F))$ is gg-closed set in $(X, \tau)$. But $(gof)^{-1}(F)=f^{-1}(g^{-1}(F))$ and hence $gof$ is gg-irresolute.

(ii) Let $F$ be any gg-closed set in $(Z, \eta)$. Since $g$ is gg-continuous then $g^{-1}(F)$ is gg-closed set in $(Y, \sigma)$. Since $f$ is gg-irresolute $f^{-1}(g^{-1}(F))$ is gg-closed set in $(X, \tau)$. But $(gof)^{-1}(F)=f^{-1}(g^{-1}(F))$ and hence $gof$ is gg-continuous.

Theorem 3.52: If $f: (X, \tau) \rightarrow (Y, \sigma)$ is strongly gg-continuous then $f$ is continuous but converse is not true.

Proof: Let $f: X \rightarrow Y$ be strongly gg-continuous. Let $F$ be any closed set in $Y$. Since every closed set is gg-closed and hence $F$ is gg-closed set in $Y$. Since $f$ is strongly gg-continuous then $f^{-1}(F)$ is closed set in $X$. Therefore $f$ is continuous.

Example 3.53: Let $X=\{p, q, r\}$. Let $\tau=\{\varnothing, X, \{p\}, \{q\}, \{p, q\}\}$ and $\sigma=\{\varnothing, Y, \{p\}, \{q\}, \{p, q\}, \{p, r\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by $f(p)=p, f(q)=b, f(r)=r$ is continuous but not strongly gg-continuous as the gg-closed set $F=\{p, q\}$ in $Y$, $f^{-1}(F)=\{q\}$ is not a closed set in $X$. 

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Theorem 3.54 Every strongly gg-continuous is strongly g-continuous but not conversely.

Proof: Let \( f: X \to Y \) be strongly gg-continuous. Let \( F \) be any g-closed set in \( Y \). Since every g-closed set is gg-closed and hence \( F \) is gg-closed set in \( Y \). Since \( f \) is strongly gg-continuous then \( f^{-1}(F) \) is closed set in \( X \) and hence g-closed set in \( X \). Therefore \( f \) is g-continuous.

Example 3.55 In example 3.53 \( f \) is strongly g-continuous but not a strongly gg-continuous as the gg-closed set \( F=\{p, q\} \) in \( Y \), \( f^{-1}(F)=\{q\} \) is not a closed set in \( X \).

Theorem 3.56 If a mapping \( f: (X, \tau) \to (Y, \sigma) \) is strongly gg-continuous if and only if \( f^{-1}(U) \) is open in \( X \) for every gg-open set \( U \) in \( Y \).

Proof: Suppose that \( f: X \to Y \) is strongly gg-continuous. Let \( U \) be any gg-open set in \( Y \) and hence \( U^c \) is gg-closed set in \( Y \). Since \( f \) is strongly gg-continuous, \( f^{-1}(U) \) is closed set in \( X \). But \( f^{-1}(U^c) = X \setminus f^{-1}(U) \). Thus \( f^{-1}(U) \) is open in \( X \).

Conversely

Suppose that \( f^{-1}(U) \) is open set in \( X \) for every gg-open set \( U \) in \( Y \). Let \( F \) be any gg-closed set in \( Y \) and hence \( F^c \) is gg-open in \( X \). But \( f^{-1}(F^c) = X \setminus f^{-1}(F) \). Thus \( X \setminus f^{-1}(F) \) is open in \( X \) and so \( f^{-1}(F) \) is closed in \( X \). Therefore \( f \) is strongly gg-continuous.

Theorem 3.57 Every strongly continuous is strongly gg-continuous but not conversely.

Proof: Let \( f: X \to Y \) is strongly continuous. Let \( G \) be any gg-open set in \( Y \) and also any subset of \( Y \). Since \( f \) is strongly continuous then \( f^{-1}(G) \) is both open and closed in \( X \), say \( f^{-1}(G) \) is open in \( X \). Therefore \( f \) is strongly gg-continuous.

Example 3.58 Let \( X=Y=\{p, q, r\} \). Let \( \tau=\{\varnothing, X, \{p\}, \{q\}, \{p, q\}, \{p, r\}\} \) and \( \sigma=\{\varnothing, Y, \{p\}, \{q\}, \{p, q\}\} \). Let \( f: (X, \tau) \to (Y, \sigma) \) be a function defined by \( f(p)=p, f(q)=r, f(r)=p \) is strongly gg-continuous but not a strongly continuous as the set \( F=\{p\} \) in \( Y \), \( f^{-1}(F)=\{r\} \) is not a clopen set in \( X \).

Theorem 3.59 Every strongly gg-continuous is gg-continuous but not conversely.

Proof: Let \( f: X \to Y \) be strongly gg-continuous. Let \( F \) be any closed set in \( Y \) and hence gg-closed in \( Y \). Since \( f \) is strongly gg-continuous then \( f^{-1}(F) \) is closed set in \( X \) and hence gg-closed set in \( X \). Therefore \( f \) is gg-continuous.

Example 3.60 In example 3.53, \( f \) is gg-continuous but not strongly gg-continuous as the gg-closed set \( F=\{p, q\} \) in \( Y \), \( f^{-1}(F)=\{p, q\} \) is not a closed set in \( X \).

Theorem 3.61 In discrete topological space, every strongly gg-continuous is strongly continuous.

Proof: Let \( f: X \to Y \) be strongly gg-continuous in a discrete topological space. Let \( F \) be any subset of \( Y \). Since \( F \) is both open and closed subset of \( Y \) in discrete space. We have the following two cases.

Case (i) Let \( F \) be any closed subset of \( Y \) and hence gg-closed in \( Y \). Since \( f \) is strongly gg-continuous then \( f^{-1}(F) \) is closed in \( X \).

Case (ii) Let \( F \) be any open subset of \( Y \) and hence gg-open in \( Y \). Since \( f \) is strongly gg-continuous then \( f^{-1}(F) \) is open in \( X \).

Therefore \( f^{-1}(F) \) is both open and closed in \( X \). Hence \( f \) is strongly continuous.

Theorem 3.62 Let \( f: X \to Y \) and \( g: Y \to Z \) be any two functions. Then

i) \( \text{gof}: X \to Z \) is strongly gg-continuous if both \( f \) and \( g \) are gg-continuous.

ii) \( \text{gof}: X \to Z \) is strongly gg-continuous if \( g \) is strongly gg-continuous and \( f \) is continuous.

iii) \( \text{gof}: X \to Z \) is gg-irresolute if \( g \) is strongly gg-continuous and \( f \) is gg-continuous.

iv) \( \text{gof}: X \to Z \) is continuous if \( g \) is gg-continuous and \( f \) is strongly gg-continuous.

Proof:

i) Let \( G \) be gg-closed set in \( (Z, \eta) \). Since \( g \) is strongly gg-continuous then \( g^{-1}(G) \) is closed set in \( (Y, \sigma) \) and hence gg-closed set in \( (Y, \sigma) \). Since \( f \) is also strongly gg continuous then \( f^{-1}(g^{-1}(G)) \) closed set in \( (X, \tau) \). But \( (\text{gof})^{-1}(G) = f^{-1}(g^{-1}(G)) \) and hence gof is strongly gg-continuous.
iv) Let $G$ be any closed set in $(Z, \eta)$. Since $g$ is strongly $gg$-continuous then $g^{-1}(G)$ is closed set in $(Y, \sigma)$. Since $f$ is strongly $gg$-continuous then $f^{-1}(g^{-1}(G))$ is $gg$-closed set in $(X, \tau)$. But $(gof)^{-1}(G) = f^{-1}(g^{-1}(G))$ and hence $gof$ is perfectly $gg$-continuous.

**Theorem 3.63** Let $f: X \to Y$ and $g: Y \to Z$ be any two functions. Then

i) $gof: X \to Z$ is strongly $gg$-continuous if $g$ is perfectly $gg$-continuous and $f$ is continuous.

ii) $gof: X \to Z$ is perfectly $gg$-continuous if $g$ is strongly $gg$-continuous and $f$ is perfectly $gg$-continuous.

**Proof:**

i) Let $G$ be any $gg$-open set in $(Z, \eta)$. Since $g$ is perfectly $gg$-continuous then $g^{-1}(G)$ is clopen set in $(Y, \sigma)$, say $g^{-1}(G)$ is open set in $(Y, \sigma)$. Since $f$ is continuous then $f^{-1}(g^{-1}(G))$ is open set in $(X, \tau)$. Thus $(gof)^{-1}(G) = f^{-1}(g^{-1}(G))$, Hence $gof$ is strongly $gg$-continuous.

ii) Let $G$ be a $gg$-open set in $(Z, \eta)$. Since $g$ is strongly $gg$-continuous then $g^{-1}(G)$ is open set in $(Y, \sigma)$. Since $f$ is perfectly $gg$-continuous then $f^{-1}(g^{-1}(G))$ clopen set in $(X, \tau)$. But $(gof)^{-1}(G) = f^{-1}(g^{-1}(G))$. Hence $gof$ is perfectly $gg$-continuous.

**Theorem 3.64** Let $(X, \tau)$ be a discrete topological space and $(Y, \sigma)$ be any topological space. Let $f: (X, \tau) \to (Y, \sigma)$ be a function. Then the following statements are equivalent.

i) $f$ is strongly $gg$-continuous

ii) $f$ is perfectly $gg$-continuous

**Proof:**

(i) $\Rightarrow$ (ii)

Let $G$ be any open set in $(Y, \sigma)$. Since $f$ is strongly $gg$-continuous then $f^{-1}(G)$ is open set in $(X, \tau)$. But in discrete space, $f^{-1}(G)$ is closed set in $(X, \tau)$. Thus $f^{-1}(G)$ is both open and closed in $(X, \tau)$. Hence $f$ is perfectly $gg$-continuous.

(ii) $\Rightarrow$ (i)

Let $U$ be any $gg$-open set in $(Y, \sigma)$. Since $f$ is perfectly continuous then $f^{-1}(G)$ is both open and closed in $(X, \tau)$. Hence $f$ is strongly $gg$-continuous.

**Theorem 3.65** Let $(X, \tau)$ be any topological space and $(Y, \sigma)$ be $ggT_c$ space and $f: (X, \tau) \to (Y, \sigma)$ be a map. Then the following are equivalent.

i) $f$ is strongly $gg$-continuous

ii) $f$ is continuous

**Proof:**

(i) $\Rightarrow$ (ii)

Let $F$ be any closed set in $(Y, \sigma)$. Since every closed set is $gg$-closed and hence $F$ is $gg$-closed in $(Y, \sigma)$. Since $f$ is strongly $gg$ continuous then $f^{-1}(F)$ is closed set in $(X, \tau)$. Hence $f$ is continuous.

(ii) $\Rightarrow$ (i)

Let $G$ be any $gg$-closed set in $(Y, \sigma)$. Since $(Y, \sigma)$ is $ggT_c$ space, $F$ is closed set in $(Y, \sigma)$. Since $f$ is continuous then $f^{-1}(G)$ is closed set in $(X, \tau)$. Hence $f$ is strongly $gg$-continuous.

**Theorem 3.66** Let $f: (X, \tau) \to (Y, \sigma)$ be a map. Both $(X, \tau)$ and $(Y, \sigma)$ are $ggT_c$ space. Then the following are equivalent.

i) $f$ is $gg$-irresolute

ii) $f$ is strongly $gg$-continuous

iii) $f$ is continuous

iv) $f$ is $gg$-continuous

The proof is obvious.

**Theorem 3.67** Let $X$ and $Y$ be $ggT_\infty$ spaces. Then for the function $f: (X, \tau) \to (Y, \sigma)$ the following are equivalent.

i) $f$ is $gg$-irresolute
ii) $f$ is $gg$-irresolute

Proof:

(i) $\rightarrow$ (ii)

Let $f : X \rightarrow Y$ be $gc$-irresolute. Let $F$ be a $g$-closed set in $Y$ and hence $gg$-closed in $Y$. Since $f$ is $gc$-irresolute then $f^{-1}(F)$ is $g$-closed set in $X$ and hence $gg$-closed set in $X$. Therefore $f$ is $gg$-irresolute.

(i) $\rightarrow$ (ii)

Let $f : X \rightarrow Y$ be $gg$-irresolute. Let $F$ be a $g$-closed set in $Y$ and hence $gg$-closed in $Y$. Since $f$ is $gg$-irresolute then $f^{-1}(F)$ is $gg$-closed set in $X$. But $X$ is $ggT_{gc}$ space and hence $f^{-1}(F)$ is $g$-closed set in $X$. Therefore $f$ is $g$-irresolute.

4. CONCLUSION

In this paper we introduced and studied the basic properties of $gg$-Continuous functions, strongly $gg$-continuous functions and $gg$-irresolute maps in topological space $(X, \tau)$. Also we studied the relation between $gg$-continuous functions and existing continuous functions in topological space. Further we will introduce and study basic properties of $gg$-closed maps and $gg$-open maps in topological space $(X, \tau)$.

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6. REFERENCES


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