AN INTRODUCTION TO LATTICE ORDERED G-MODULES

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Abstract: Lattice theory is a branch of Mathematics which is fast making inroads into various mathematical disciplines. The concepts of lattice ordered groups and rings emerged as a result of the application of lattice theory in algebra. The structure of $G$-modules is an effective tool in algebra which is mainly used in representation theory. In this paper we introduce the notion of lattice ordered $G$-modules which would facilitate an advanced research in the area of lattice ordered groups and other algebraic systems.

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INTRODUCTION:

Group theory is a fundamental concept in algebra and has a wide range of applications in modern mathematics[8, 11]. Representation theory is an important branch in mathematics which deals with the study of group theoretical problems and associative algebras. The representation theory of finite groups proposed by G.Frobenius in 1896 was a major breakthrough in the development of this subject. Many researchers have worked in this area and have evolved a strong framework of the concept. The main idea in this approach is to study a group in terms of simple linear algebraic tools like linear transformations of vector spaces and modules. The study of $G$-modules and their properties is of great importance in this regard [4, 5, 6]. We have done a good deal of research in the area of $G$-modules and have introduced some novel concepts in $G$-modules particularly in the context of fuzzy and rough set theories[12, 13, 21].

The concept of ordering is an inevitable tool in the study of mathematics. It led to the origin of lattice theory which has become an integral part of mathematics[9, 10, 20]. The concepts of lattices and many algebraic structures have been combined together by many researchers to form various ordered algebraic structures which include ortho modular posets, Boolean algebras, relation algebras, lattice ordered groups, lattice ordered rings and so on. The structure of lattice ordered groups and their properties have been studied in [2, 3, 7, 14, 16, 17]. Lattice ordered rings,fields and Riesz spaces are discussed in detail in [1, 15, 16, 18, 19, 22].

The aim of this paper is to introduce a new ordered algebraic structure namely lattice ordered $G$-modules and to discuss some of the basic related results. In section 2 we recall some of the fundamentals of lattices, lattice ordered groups and vector lattices. In section 3 we define a lattice ordered $G$-module and prove some related results. We conclude in section 4 with possible research work that can be done further in the area of lattice ordered $G$-modules.

PRELIMINARIES:

In this section, we see some basic definitions that will be needed in the sequel.

Definition 2.1 [20] A lattice is a partially ordered set $(L, \leq)$ in which every pair of elements $a, b \in L$ has a greatest lower bound $a \wedge b$ (called their meet) and a least upper bound $a \vee b$ (called their join). Here $\leq$ is the partial order on the set $L$. 
**Definition 2.2** [20] A lattice is an algebraic system \((L,\wedge,\vee)\) with two binary operations \(\wedge\) and \(\vee\) which are both commutative, associative and satisfy the absorption laws. It has been shown in [20] that the above two definitions are equivalent.

**Definition 2.3** [20] A non empty subset \(S\) of a lattice \(L\) is called a sublattice of \(L\) if \(a,b \in S \Rightarrow a \wedge b, a \vee b \in S\).

**Definition 2.4** [20] Let \(a, b \in L\). The closed interval \([a, b]\) in \(L\) is \(\{x \in L \mid a \leq x \leq b\}\).

**Definition 2.5** [2] A non-empty set \(G\) with a binary operation \(*\) and a partial order \(\leq\) is called a lattice ordered group or an \(l\)-group iff

1. \((G,\ast)\) is a group
2. \((G,\leq)\) is a lattice
3. \(g \leq h \Rightarrow a \ast g \ast b \leq a \ast h \ast b\) for all \(a, b, g, h \in G\).

**Example 2.6** [19] \((\mathbb{Z},+),(\mathbb{Q},+)\) and \((\mathbb{R},+)\) are lattice-ordered groups under the natural ordering.

**Example 2.7** [19] \(Aut(P)\) for any totally ordered set \(P\) is a lattice ordered group under the natural ordering.

It has been shown in [3] that in a lattice ordered group \(G\) the following properties hold

1. \(a \ast (g \wedge h) = (a \ast g) \wedge (a \ast h)\)
2. \(a \ast (g \vee h) = (a \ast g) \vee (a \ast h)\)
3. \(g \wedge h = h \ast (g \vee h)^{-1} \ast g; \forall a, g, h \in G\)

**Definition 2.8** [3] A subgroup \(H\) of \(G\) is called an \(l\)-subgroup of \(G\) if \(H\) is a sublattice of \(G\).

**Definition 2.9** [3] An \(l\)-subgroup \(H\) of \(G\) is called a convex \(l\)-subgroup of \(G\) if it is a convex subset of \(G\). (i.e., if \(g, h\) are two elements of \(G\) such that \(g \leq h\), then \([g, h] \subseteq G\).

**Definition 2.10** [19] The positive part of an element \(g\) in an \(l\)-group \(G\) is \(g^+ = g \vee e\). The negative part of \(g\) is \(g^- = g \wedge e\) and the modulus of \(g\) is \(|g| = g \vee g^{-1}\). \(g = g^+ \ast (g^-)^{-1}\) and \(g \geq g^+ \ast g^-\)

**Definition 2.11** [19] The set \(G^+ = \{g \in G : g \geq e\}\) is termed as the positive cone in \(G\) and its elements are termed as positive.

**Theorem 2.12** [19] (1) \(G^+\) is a semigroup with identity element \(e\).

2. \(G^+ \cap -G^+ = e\) where \(-G^+ = \{g \in G : -g \in G^+\}\)
3. \(G^+\) is a normal subset of \(G\) i.e., \(g \in G^+ \Rightarrow -h \ast g \ast h \in G^+; \forall h \in G\)

**Theorem 2.13** [19] \(G\) is an \(l\)-group iff \(G^+\) generates \(G\) and \(G^+\) is a lattice under the induced order.

**Definition 2.14** [2] Let \(G\) be an \(l\)-group. A non-empty subset \(I\) of \(G\) is called an \(l\)-ideal of \(G\) if

1. \(I\) is a subgroup of \(G\)
2. \(i, j \in I \Rightarrow i \vee j \in I\)
3. \(i \wedge j \in I, i, g \in G \Rightarrow i \wedge g \in I\).

**Definition 2.15** [2] An \(l\)-ideal of \(G\) is called an \(l\)-prime ideal of \(G\) if \(I\) is properly contained in \(G\) and \(i \wedge j \in I \Rightarrow i \in I\) or \(j \in I\).

**Definition 2.16** [2] Let \(G\) be a \(l\)-group. A non-empty subset \(F\) of \(G\) is called an \(l\)-filter of \(G\) if

1. \(F\) is a subgroup of \(G\)
2. \(i, j \in F \Rightarrow i \wedge j \in F\)
3. \(i \in F, g \in G \Rightarrow i \vee g \in F\).
Definition 2.17 [3] If $G$ and $H$ are lattice-ordered groups then by an $l$-homomorphism from $G$ to $H$ we mean a mapping $\rho : G \rightarrow H$ that is both a group homomorphism and a lattice homomorphism. ie,

(1) $\rho (g \ast h) = \rho (g) \ast \rho (h)$
(2) $\rho (g \wedge h) = \rho (g) \wedge \rho (h)$ and
(3) $\rho (g \vee h) = \rho (g) \vee \rho (h) ; \forall g, h \in G$.

Definition 2.18 [22] An ordered vector space is a real vector space $M$ which is also an ordered space with the linear and order structures connected by the implications

(1) If $x, y, z \in M$ and $x \leq y$ then $x + z \leq y + z$
(2) If $x, y \in M$ , $x \leq y$ and $0 \leq \alpha \in R$ , then $\alpha x \leq \alpha y$

Definition 2.19 [22] An ordered vector space which is also a lattice is a vector lattice or Riesz space.

Example 2.20 [22] The usual or standard order on $R^n$ is that in which $(x_1, x_2, ..., x_n) \leq (y_1, y_2, ..., y_n)$ means that $x_k \leq y_k$ for $k = 1, 2, ..., n$. This order makes $R^n$ into a vector lattice in which

$$(x_1, x_2, ..., x_n) \wedge (y_1, y_2, ..., y_n) = (x_1 \wedge y_1, x_2 \wedge y_2, ..., x_n \wedge y_n)$$
and

$$(x_1, x_2, ..., x_n) \vee (y_1, y_2, ..., y_n) = (x_1 \vee y_1, x_2 \vee y_2, ..., x_n \vee y_n)$$

Definition 2.21 [22] A vector sublattice of a vector lattice is simply a vector subspace which is also a sublattice.

LATTICE ORDERED $G$ -MODULES:

Definition 3.1 Let $G$ be an $l$-group. A vector lattice $M$ is called a lattice ordered $G$-module or an $lG$-module if for every $g \in G$ and $x \in M$ , there exists a product $g \cdot x \in M$ (called the action of $G$ on $M$ ) satisfying the following axioms

(1) $e \cdot x = x$
(2) $(g \ast h) \cdot x = g \cdot (h \cdot x)$
(3) $g \cdot (rx + sy) = r(g \cdot x) + s(g \cdot y)$
(4) $(g \cdot x) \wedge (h \cdot y) = (g \wedge h) \cdot (x \wedge y)$
(5) $g \cdot x) \vee (h \cdot y) = (g \vee h) \cdot (x \vee y) ; \forall g, h \in G ; x, y \in M ; r, s \in R$.

Every lattice ordered $G$-module can be identified as the action of the lattice ordered group $G$ on the underlying vector lattice of $M$ which is equivalent to an $l$-group homomorphism $\rho : G \rightarrow Aut(M)$ defined by sending each $g \in G$ to an automorphism $\rho_g : M \rightarrow M$ as $x \mapsto g \cdot x$.

Example 3.2 The action of $Z$ on $R^n$ is an example of a lattice ordered $G$-module.

Definition 3.3 Let $M$ be a $lG$-module. A vector sublattice $N$ of $M$ is a lattice ordered $G$-submodule or $lG$-submodule if $N$ is also a lattice ordered $G$-module under the same action of $G$.

Example 3.4 Let $G = \{-1, 1\}$. Then $R$ is an $lG$-module over $Q$ under the natural ordering and for any irrational number $'r'; Q(r)$ is an $lG$-submodule of $R$.

Definition 3.5 A $lG$-submodule $N$ of $M$ is called a convex $lG$-submodule of $M$ if it is a convex subset of $M$.

Definition 3.6 Let $M$ be a $lG$-module. A non-empty subset $I$ of $M$ is called an $G$-ideal of $M$ if

(1) $I$ is a $G$-submodule of $M$
(2) $i, j \in I \Rightarrow i \vee j \in I$
(3) $i \in I, x \in M \Rightarrow i \wedge x \in I$.

Definition 3.7 A $G$-ideal of $M$ is called a $G$-prime ideal of $M$ if $I$ is properly contained in $M$ and $a \wedge b \in I \Rightarrow a \in I$ or $b \in I$.

Definition 3.8 Let $M$ be an $lG$-module. A non-empty subset $F$ of $M$ is called a $G$-filter of $M$ if

(1) $F$ is a $G$-submodule of $M$
(2) $i, j \in F \Rightarrow i \land j \in F$

(3) $i \in F, x \in M \Rightarrow i \lor x \in F$.

**Theorem 3.9** Let $M$ be an $lG$-module where $G$ is abelian. Let $H$ be an $l$-subgroup of $G$ and consider $N = \{h \cdot x \mid h \in H; x \in M\}$ which is closed under vector addition. Then $N$ is an $lG$-submodule of $M$.

**Proof.** For any $g \in G; h, k \in H; r \in R$ and $x, y \in M$ we have $r(h \cdot x) = h \cdot (rx) \in h \cdot M \subseteq N$. Further $g \cdot (h \cdot x) = (g \cdot h) \cdot x = (h \cdot (g \cdot x)) \in h \cdot M \subseteq N$. Also $(h \cdot x) \land (k \cdot y) = (h \land k) \cdot (x \land y) \in N$. And $(h \cdot x) \lor (k \cdot y) = (h \lor k) \cdot (x \lor y) \in N$.

**Remark 3.10** $N$ is called the lattice ordered $G$-submodule of $M$ induced by $H$.

**Theorem 3.11** Let $M$ be an $lG$-module where $G$ is abelian. Let $G^*$ denote the positive cone of $G$ and consider $N = \{g \cdot x \mid g \in G^*; x \in M\}$ which is closed under vector addition. Then $N$ is an $lG$-submodule of $M$.

**Proof.** For any $g \in G; h \in G^*; r \in R$ and $x, y \in M$ we have $r(g \cdot x) = g \cdot (rx) \in g \cdot M \subseteq N$. And $g \cdot (g \cdot x) = (g \cdot g^*) \cdot x = (g \cdot (g \cdot x)) \in g \cdot M \subseteq N$. Also $(g \cdot x) \land (h \cdot y) = (g \land h) \cdot (x \land y) \in N$. And $(g \cdot x) \lor (h \cdot y) = (g \lor h) \cdot (x \lor y) \in N$.

**Definition 3.12** A lattice ordered $G$-module $M$ is called regular if for any $x, y \in M$ there exists exactly one element $g$ in the lattice ordered group $G$ such that $g \cdot x = y$.

**Theorem 3.13** Let $M$ be a regular $lG$-module where $G$ is abelian. Let $G^*$ be a convex $l$-subgroup of $G$ and for any $x \in M$ consider $N_x = \{g \cdot x \mid g \in G^*\}$ which is closed under vector addition. Then $N_x$ is a convex $lG$-submodule of $M$.

**Proof.** From theorem 3.9 it is clear that $N_x$ is an $lG$-submodule of $M$. Now we will prove that it is a convex subset of $M$. Let $g \cdot x, h \cdot x \in N_x$ where $g, h \in G^*$. $[(g \cdot x \land h \cdot x), (g \cdot x \lor h \cdot x)] = [(g \land h) \cdot x, (g \lor h) \cdot x] = \{m \in M \mid m = g \cdot x \}$ such that $g \land h \leq g \lor h \leq g \cdot x$. Since $G^*$ is convex, we have $\{g \land h, g \lor h\} \subseteq G^*$. Therefore $[g \cdot x \land h \cdot x, (g \cdot x \lor h \cdot x)] \subseteq N_x$. Thus $N_x$ is a convex $lG$-submodule of $M$.

**Theorem 3.14** Let $I$ be an $l$-ideal of $G$. Then the $lG$-submodule induced by $I$ is an $G$-ideal of $M$ if $M$ is transitive, i.e. (if for any two elements $x, y \in M$ there exists an element $g \in G$ such that $g \cdot x = y$).

**Proof.** Consider $N = \{i \cdot x \mid i \in I; x \in M\}$. Let $i, j \in I$ and $x, y, z \in M$. $(i \cdot x) \land (j \cdot y) = (i \land j) \cdot (x \land y) \in N$. Since $M$ is transitive there exists an element $g \in G$ such that $y = g \cdot z$ so that $(i \cdot x) \land y = (i \cdot x) \land (g \cdot z) = (i \land g) \cdot (x \land z) \in N$.

Similarly we can prove that if $M$ is a transitive $lG$-module and $J$ is an $l$-filter of $G$ then the $lG$-submodule induced by $J$ is a $G$-filter of $M$.

**Corollary 3.15** Let $M$ be a transitive $lG$-module and $I$ an $l$-prime ideal of $G$. Then the $lG$-submodule induced by $I$ is a $G$-prime ideal of $M$.

**Proof.** Consider $N = \{i \cdot x \mid i \in I; x \in M\}$. Clearly $N$ is a $G$-ideal of $M$. Let $n_1 \land n_2 \in N$ then there exists an $i_1 \in I$ and $x_1 \in M$ such that $n_1 \land n_2 = i_1 \cdot x_1$. Since $M$ is transitive there exists $g_1, g_2 \in G$ such that $n_1 = g_1 \cdot x_1$ and $n_2 = g_2 \cdot x_1$. Now $n_1 \land n_2 \in N \Rightarrow (g_1 \cdot x_1) \land (g_2 \cdot x_1) \in N \Rightarrow (g_1 \land g_2) \cdot x_1 \in N \Rightarrow g_1 \land g_2 \in I \Rightarrow g_1 \in I$ or $g_2 \in I \Rightarrow g_1 \land x_1 \in N$ or $g_2 \land x_1 \in N \Rightarrow n_1 \in N$ or $n_2 \in N$. Thus $N$ is a $G$-prime ideal of $M$.

**Theorem 3.16** Let $M$ be a lattice ordered $G$-module. Fix any $x \in M$ and let $G_x = \{g \in G \mid g \cdot x = x\}$. Then $G_x$ is a $l$-subgroup of $G$.
Proof. It has been shown in [8] that $G_\ast$ is a subgroup of $G$. Now let $g, h \in G_\ast$. Then $(g \land h) \cdot x = (g \cdot x) \land (h \cdot x) = x \land x = x$. Hence $g \land h \in G_\ast$. Similarly we can show that $g \lor h \in G_\ast$. Hence $G_\ast$ is a $l$-subgroup of $G$.

Remark 3.17 $G_\ast$ is called the $l$-isotropy subgroup of $G$.

Corollary 3.18 $G_\ast$ is an $l$-ideal of $G$ if $\forall g \in G; g \cdot x \geq x$ and $G_\ast$ is a $l$-filter of $G$ if $\forall g \in G; g \cdot x \leq x$.

Theorem 3.19 Let $M$ be a lattice ordered $G$-module and let $M^G = \{ x \in M \mid g \cdot x = x; \forall g \in G \}$ denote the set of all fixed points. Then $M^G$ is a lattice ordered $G$-submodule of $M$.

Proof. Let $x, y \in M^G$. Then $g \cdot (x + y) = (g \cdot x) + (g \cdot y) = x + y$. For any $r \in R; g \cdot (rx) = r(g \cdot x) = rx$. For any $h \in G$ we have $g \cdot (h \cdot x) = g \cdot x = x g \cdot (x \land y) = (g \cdot x) \land (g \cdot y) = x \land y$ which implies that $x \land y \in M^G$. Similarly we can prove that $x \lor y \in M^G$. Hence $M^G$ is a lattice ordered $G$-submodule of $M$.

Remark 3.20 $M^G$ is called the $l$-stabilizer $G$-submodule of $M$.

Theorem 3.21 The $l$-orbit of any $x \in M$ defined by $O_x = \{ g \cdot x \in M \mid g \in G \}$ is a sublattice of $M$.

Proof. Let $g \cdot x, h \cdot x \in O_x$. Then $(g \cdot x) \land (h \cdot x) = (g \land h) \cdot x \in O_x$. Also $(g \cdot x) \lor (h \cdot x) = (g \lor h) \cdot x \in O_x$. Hence $O_x$ is a sublattice of $M$.

Definition 3.22 The $G$-annihilator of an element $x \in M$ is defined as $G_a(x) = \{ g \in G \mid g \cdot x = 0 \}$ where 0 is the identity element of $M$.

Theorem 3.23 For any $x \in M$, $G_a(x)$ is a $l$-subgroup of $G$.

Proof. Let $g, h \in G_a(x)$. $(g \ast h) \cdot x = g \cdot (h \cdot x) = g \cdot 0 = e_M$. And $g^{-1} \cdot x = (g \cdot x)^{-1} = 0$. Also $(g \land h) \cdot x = g \cdot x \land h \cdot x = 0$ and $(g \lor h) \cdot x = g \cdot x \lor h \cdot x = 0$. Thus $G_a(x)$ is an $l$-subgroup of $G$.

Corollary 3.24 $G_a(x)$ is an $l$-ideal of $G$ if $g \cdot x \in M^G; 0.10 cm \forall g \in G$.

Definition 3.25 The $G$-annihilator of a lattice ordered $G$-module $M$ is the intersection of the $G$-annihilators of its elements.

Remark 3.26 The $G$-annihilator of $M$ is an $l$-ideal of $G$ if $g \cdot M \subseteq M^G; \forall g \in G$.

Definition 3.27 An element $g \in G^+$ is called a $f$-element of $G$ on $M$ if $\forall x, y \in M; x \land y = 0 \Rightarrow g \cdot x \land y = 0$.

Theorem 3.28 Let $f(M)$ denote the set of all $f$-elements, then

1. $f(M)$ is a convex sub semigroup of $G^+$
2. $f(M)$ is a sublattice of $G^+$.

Proof. Let $g, h \in f(M)$. Then for any $x, y \in M$ such that $x \land y = e_M$ we have $(h \cdot x) \land y = 0 \Rightarrow (g \cdot (h \cdot x)) \land y = 0 \Rightarrow ((g \ast h) \cdot x) \land y = 0 \Rightarrow g \ast h \in f(M)$. Hence $f(M)$ is a sub semigroup of $G^+$. Suppose that $g \leq h$. Then let $k \in [g, h]$. Thus $g \leq k \leq h \Rightarrow g \cdot x \leq k \cdot x \leq h \cdot x \Rightarrow (g \cdot x) \land y \leq (k \cdot x) \land y \leq (h \cdot x) \land y \Rightarrow 0 \leq (k \cdot x) \land y \leq 0 \Rightarrow (k \cdot x) \land y = 0$.
which shows that \( k \in f(M) \). Hence \( f(M) \) is a convex subsemigroup of \( G^+ \). Now \( (g \cdot x) \wedge y = 0 \) and \((h \cdot x) \wedge y = 0 \) \( \Rightarrow ((g \cdot x) \wedge y) \wedge ((h \cdot x) \wedge y) = 0 \) \( \Rightarrow ((g \cdot x) \wedge (h \cdot x)) \wedge y = 0 \) \( \Rightarrow (g \wedge h) \cdot x \wedge y = 0 \) \( \Rightarrow g \wedge h \in f(M) \). Similarly we can show that \( g \vee h \in f(M) \). Thus \( f(M) \) is a sublattice of \( G^+ \).

**Theorem 3.29** Let \((G, \leq)\) be an \( l\)-group and \( M \) be a regular \( G \)-module. Then \( G \) induces an ordering on \( M \) under which \( M \) is a lattice ordered \( G \)-module.

**Proof.** Let \( x, y \in M \). Since \( M \) is regular; for any element \( a \notin M \) there exist unique elements \( g, h \in G \) such that \( x = g \cdot a \) and \( y = h \cdot a \). Define a binary relation \( \leq \) on \( M \) by \( x \leq y \) if \( g \leq h \) in \( G \). Since \( g \leq g \) in \( G \) we have \( x \leq x \) in \( M \). Hence \( \leq \) is reflexive. Now let \( x \leq y \) and \( y \leq x \) in \( M \) then \( g \leq h \) and \( h \leq g \) in \( G \) which implies that \( g = h \) so that we have \( x = y \). Hence \( \leq \) is anti-symmetric. Also we can show that \( \leq \) is transitive. Thus \( \leq \) is a partial order on \( M \). Also \( x \wedge y = (g \cdot a) \wedge (h \cdot a) = (g \wedge h) \cdot a \) and \( x \vee y = (g \cdot a) \vee (h \cdot a) = (g \vee h) \cdot a \). Thus \( M \) is a lattice ordered \( G \)-module under \( \leq \). This ordering is called the ordering induced by \( a \).

**CONCLUSION:**

In this paper we have introduced the notion of lattice ordered \( G \)-modules and studied some of its properties. We believe that this will surely facilitate further research in the area of lattice ordered groups and representation theory.

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