STUDY OF FIXED POINT THEOREM FOR BANACH OPERATOR PAIR OF T-CONTRACTION MAPPINGS IN CONE METRIC SPACES

Surendra Kumar Tiwari¹ and Kaushik Das²

¹Department of Mathematics, Dr. C.V. Raman University, Kota, Bilaspur (C.G.)-India
²Department of Mathematics, Gobardanga Hindu College, Habra, West Bangal- India

Email: sk10tiwari@gmail.com

Abstract: The purpose of this paper is to establish and extend the concept of T-Rhoades contraction to the case of Banach operator pairs of mappings in the setting of complete cone metric spaces without the assumption of normality condition.

Keywords: Cone metric space, complete cone metric space, fixed point, common fixed point, T- Rhoades contraction, Banach operator pair.

I. INTRODUCTION

The first fundamental result of fixed point theory is Banach contraction principle, which introduced in 1922 by Banach [1] as the following theorem:-

**Theorem 1.** Let \((X, d)\) be a complete metric space and let \(T: X \rightarrow X\) be Banach contraction mapping, if there exist a constant \(a \in [0,1)\) such that

\[
d(Tx, Ty) \leq ad(x, y), \text{ for all } x, y \in X.
\]

Then \(T\) has a unique fixed point. It is one of the famous and traditional theorems in modern mathematics which is widely applied in many other branches of science and applied science.

In 1968 and 1969, Kannan [2, 3] introduced the concept of Kannan mappings as follows:-**Theorem 2.** Let \((X, d)\) be a complete metric space and let \(T: X \rightarrow X\) be Kannan contraction mapping, if there exist a constant \(b \in [0, \frac{1}{2})\) such that

\[
d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)], \text{ for all } x, y \in X.
\]

Then \(T\) has a unique fixed point.

Chatterjea [4], introduced the concept of chatterjea contraction mapping in 1972, as follows:-**Theorem 3.** Let \((X, d)\) be a complete metric space and let \(T: X \rightarrow X\) be Chatterjea contraction mapping, if there exist a constant \(c \in [0, \frac{1}{2})\) such that

\[
d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)], \text{ for all } x, y \in X.
\]
Then T has a unique fixed point.

In 1972, Zamfirescu [5], introduced the concept of Zamfirescu mapping as follows:

**Theorem 4.** Let \((X, d)\) be a complete metric space and let \(T : X \to X\) be a Zamfirescu contraction mapping, if there exist a constant \(\alpha \in [0, 1), \beta \in \left[0, \frac{1}{2}\right)\) and \(\gamma \in \left[0, \frac{1}{2}\right)\) such that at least one of the following conditions is true.

\[
\begin{align*}
(z_1) \quad &d(Tx, Ty) \leq \alpha d(x, y), \\
(z_2) \quad &d(Tx, Ty) \leq \beta [d(x, Tx) + d(y, Ty)], \\
(z_3) \quad &d(Tx, Ty) \leq \gamma [d(x, Ty) + d(y, Tx)],
\end{align*}
\]

for all \(x, y \in X\).

Then \(T\) has a unique fixed point. In the same way, the Banach contraction principle provides a constructive method of finding a unique solution for models involving various types of differential and integral equations. This principle has been studied and generalized by several authors in various directions in the literature [6-18].

In 2007, Huang and Zhang [19] introduced the notion of Cone metric spaces. They replaced the real number system by ordered Banach space and showed some fixed point theorems of different type of contractive mappings on cone metric spaces. Subsequently, many authors generalized and studied fixed and common fixed point results in cone metric spaces for normal and non-normal cone see for instance ([20-45]). Afterwards, Subramanian [46] gave introduced and called Banach operator of type k and obtained the fixed point in complete metric space. Recently, Chen and Li [47] extended the concept of Banach operator of type k to Banach operator pair and proved various best approximation results using common fixed point theorems for f- non expansive mappings. Al-thagafi and Shahzad [48] and Hussain [51] generalized the results of Chen and Li [47]. In [49], authors have proved some common fixed point theorems for a Banach pair of mapping satisfying T-Hardy Rogers type contraction condition in cone metric spaces. In sequel, Ozturk and Basarir [53], proved some common fixed point theorems f- contraction mappings in cone metric spaces without the assumption of normality condition of the cone. In 2014, Dubey et al [52] generalized the results of [49] and proved some common fixed point theorems for generalized T-Hardy Rogers contraction condition in cone metric spaces to the case of Banach operator pair. In sequel, Raghvendra et al. [50] have proved common fixed point theorems for two Banach pairs of mapping which satisfying contraction conditions in cone metric spaces.

The aim of this paper is to prove common fixed point theorems for two Banach pair of mappings which satisfying T-Rhoades contraction conditions in cone metric spaces without the assumption of normality condition of the cone.

II. PRELIMINARY NOTES

First, we recall some standard notations and definitions which we needed them in the sequel.
Definition 2.1([17]): Let \( E \) be a real Banach space and \( P \) be a subset of \( E \) and 0 denote to the zero element in \( E \), then \( P \) is called a cone if and only if:

(i) \( P \) is a non-empty set closed and \( P \neq \{0\} \),
(ii) If \( a, b \) are non-negative real numbers and \( x, y \in P \), then \( ax + by \in P \),
(iii) \( x \in P \ and \ -x \in P \implies x = 0 \iff P \cap (-P) = \{0\} \).

Given a cone \( P \subset E \), we define a partial ordering \( \leq \) on \( E \) with respect to \( P \) by \( x \leq y \) if and only if \( y - x \in \text{int} \( P \) \) (where \( \text{int} \( P \) \) denotes the interior of \( P \)). If \( \text{int} \( P \) \neq \emptyset \), then cone \( P \) is solid. The cone \( P \) called normal if there is a number \( K > 0 \) such that for all \( x, y \in E \),

\[
0 \leq x \leq y \implies \| x \| \leq k \| y \| .
\]

The least positive number \( k \) satisfying the above is called the normal constant of \( P \).

Definition 2.2([17]): Let \( X \) be a non-empty set. Suppose \( E \) is a real Banach space, \( P \) is a cone with \( \text{int} \( P \) \neq \emptyset \) and \( \leq \) is a partial ordering with respect to \( P \). If the mapping \( d: X \times X \to E \) satisfies

(i) \( 0 < d(x,y) \) for all \( x, y \in X \) and \( (x, y) = 0 \) if and only if \( x = y \),
(ii) \( d(x,y) = d(y,x) \) for all \( x, y \in X \),
(iii) \( d(x,y) \leq d(x,z) + d(z,y) \) for all \( x, y, z \in X \).

Then \( d \) is called a cone metric on \( X \), and \( (X, d) \) is called a cone metric space. The concept of cone metric space is more general than that of a metric space.

Example 2.3: Let \( E = \mathbb{R}^2 \), \( P = \{ (x,y) \in E : x, y \geq 0 \} \), \( X = \mathbb{R} \) and \( d: X \times X \to E \) defined by \( d(x,y) = (|x - y|, \alpha |x - y|) \), where \( \alpha \geq 0 \) is a constant. Then \( (X, d) \) is a cone metric space.

Definition 2.4([29]): Let \( (X, d) \) be a cone metric space, \( x \in X \) and \( \{x_n\}_{n \geq 1} \) be a sequence in \( X \). Then,

1. \( \{x_n\}_{n \geq 1} \) Converges to \( x \) whenever for every \( c \in E \) with \( \theta \ll c \), if there is a natural number \( N \) such that \( d(x_n, x) \ll c \) for all \( n \geq N \). We denote this by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \) as \( n \to \infty \)
2. \( \{x_n\}_{n \geq 1} \) is said to be a Cauchy sequence if for every \( c \in E \) with \( \theta \ll c \), if there is a natural number \( N \) such that \( d(x_n, x_m) \ll c \) for all \( n, m \geq N \).
3. \( (X, d) \) is called a complete cone metric space if every Cauchy sequence in \( X \) is Convergent.
**Definition 2.5:** A self-mapping $T$ of a metric space $(X, d)$ is said to be contraction mapping. If there exist a real number $0 \leq k < 1$ such that for all $x, y \in X$.

$$d(Tx, Ty) \leq kd(x, y).$$

The following definition is given by Beiranvand et al. [16].

**Definition 2.6([16]):** Let $T$ and $f$ be any two self-mapping of a metric space $(X, d)$. The self mapping $f$ of $X$ is said to be $T$- contraction, if there exist a real number $0 \leq k < 1$ such that

$$d(Tfx, Tfy) \leq kd(Tx, Ty) \text{ for all } x, y \in X.$$ 

If $T= I$, the identity mapping, then the definition 2.6 reduce to Banach contraction mapping.

**Example 2.7:** Let $X = [0, \infty)$ be with the usual metric. Let define two Mappings $T, f : X \to X$ as

$$fx = \beta x, \beta > 1$$

$$Tx = \frac{\alpha}{x^2}, \alpha \in R.$$ 

It is clear that $f$ is not contraction but $f$ is $T$- contraction, Since

$$d(Tfx, Tfy) \leq \left|\frac{\alpha}{\beta x^2} - \frac{\alpha}{\beta y^2}\right| = \frac{1}{\beta} |Tx - Ty|.$$

**Definition 2.8 ([16]):** Let $(X, d)$ be a metric space, and let $T : X \to X$ be self-mapping in $X$. Then

(i) $T$ is said to be sequentially convergent if the sequence $\{y_n\}$ in $X$ is convergent whenever $\{Ty_n\}$ is also convergent.

(ii) $T$ is said to be sub sequentially convergent, if $\{y_n\}$ has a convergent whenever $\{Ty_n\}$ is Convergent.

**Definition 2.9([41]):** Let $T$ be a self mapping of a normed space $X$. Then $T$ is called a Banach operator of type $k$ if

$$||T^2x - Tx|| \leq k||Tx - x||, \text{ for some } k \geq 0 \text{ and for all } x \in X.$$ 

This concept was introduced by Subrahmanyam[41], then Chen and Li[42] extended this as following:

**Definition 2.10([42]):** Let $T$ and $f$ be any two self mapping of a non empty subset $M$ of a normed space $X$. Then $(T, f)$ is a Banach operator pair, if any one of the following conditions is satisfied:

(i). $T(F(f)) \subseteq F(f)$ i.e $F(f)$ is $T$-invariant.

(ii). $fTx = Tx$ for each $x \in F(f)$.

(iii). $fTx = Tfx$ for each $x \in F(f)$.

(iv). $||Tfx - fx|| \leq k||fx - x||$ for some $k \geq 0$.

The following corollary of Rezapour[50] will be needed in the sequel.

**Corollary 2.11([50]):** Let $a, b, c, u \in E$, The real Banach space.

(i). If $a \leq b$ and $b \ll c$, then $a \ll c$.

(ii). If $a \ll b$ and $b \ll c$, then $a \ll c$. 
(iii). \(0 \leq u \ll c\) for each \(c \in \text{int} P\), then \(u = 0\).

**Remark 2.12([49]):** If \(c \in \text{int} P, 0 \leq a_n\) and \(a_n \to 0\), then there exist \(n_0\) such that \(a_n \ll c\) for all \(n > n_0\).

**Definition 2.13:** Let \((X, d)\) be cone metric spaces and let \(T, S: X \to X\) be any two functions. Then a mapping \(S\) is said to be \(T\)-Rhoades contraction, if there exist \(\alpha + \beta + \gamma + \delta < 1\) such that
\[
d(TSx, TSY) \leq \alpha d(Tx, TSy) + \beta d(Ty, TSx) + \gamma d(Tx, Ty) + \delta [d(Tx, TSx) + d(Ty, TSy)]. 
\] …. (1)

### III. Main Results.

The following main result

**Theorem 3.1:** Let \((X, d)\) be cone metric spaces and let \(T, R, S: X \to X\) be any three continuous self-mappings on \(X\). Assume that \(T\) is an injective maps. If the mapping \(T, R\) and \(S\) satisfy the condition
\[
d(TSx, TSY) \leq \alpha d(Tx, TSy) + \beta d(Ty, TSx) + \gamma d(Tx, Ty) + \delta [d(Tx, TSx) + d(Ty, TSy)]...
\] (2)
for all \(x, y \in X\). Where \(\alpha, \beta, \gamma, \delta\) are all negative constants such that \(\alpha + \beta + \gamma + \delta < 1\) Then \(R\) and \(S\) have an unique common fixed point in \(X\). Moreover, if \((T, R)\) and \((T, S)\) are Banach pair, then \(T, R\) and \(S\) have a unique common fixed point in \(X\).

**Proof:** Let \(x_0\) be an arbitrary point in \(X\). We define the iterative sequence \(\{x_{2n}\}\) and \(\{x_{2n+1}\}\) by
\[
x_{2n+1} = Rx_{2n} = R^{2n}x_0 \quad \text{……….. (3) and} \quad x_{2n+2} = Sx_{2n+1} = S^{2n+1}x_0 \quad \text{……….. (4)}.
\]

Then from (2) we have
\[
d(Tx_{2n+1}, Tx_{2n}) = d(TRx_{2n}, TSx_{2n-1})
\leq \alpha d(Tx_{2n}, TSy_{2n-1}) + \beta d(Tx_{2n-1}, TRx_{2n}) + \gamma d(Tx_{2n}, Tx_{2n-1})
\]
\[
+ \delta [d(Tx_{2n}, TSx_{2n-1}) + d(Tx_{2n-1}, TSY)].
\]

So,
\[
d(Tx_{2n+1}, Tx_{2n}) \leq \alpha d(Tx_{2n}, TSY) + (\alpha + \beta + \gamma + \delta) d(Tx_{2n}, Tx_{2n-1})
\]
\[
= \alpha d(Tx_{2n}, TSY) + (\alpha + \beta + \gamma + \delta) d(Tx_{2n}, Tx_{2n-1}) \quad \text{……….. (5)}
\]

Where \(\frac{(\alpha + \beta + \gamma + \delta)}{1 - (\alpha + \beta + \gamma + \delta)} = L < 1\).

Similarly, it can be show that
\[
d(Tx_{2n+3}, Tx_{2n+2}) \leq \frac{(\alpha + \beta + \gamma + \delta)}{1 - (\alpha + \beta + \gamma + \delta)} d(Tx_{2n+3}, Tx_{2n+2})
\]
\[
d(Tx_{2n+3}, Tx_{2n+2}) \leq L' d(Tx_{2n+3}, Tx_{2n+2}). \quad \text{……….. (6)}
\]

Where \(\frac{(\alpha + \beta + \gamma + \delta)}{1 - (\alpha + \beta + \gamma + \delta)} = L < 1\).
Thus, $d(Tx_{2n+1}, Tx_{2n}) \leq d(Tx_{2n}, Tx_{2n-1}) \leq \ldots \leq L^n d(Tx_1, Tx_0)$, for $n \geq 0$.

So, for $m, n \in N$ with $n > m$ we have

$$
d(Tx_{2n}, Tx_{2m}) \leq d(Tx_{2n}, Tx_{2n-1}) + d(Tx_{2n-1}, Tx_{2n-2}) + \ldots + d(Tx_{2m+1}, Tx_{2m})
$$

$$
\leq (L^{2n-1} + L^{2n-2} + \ldots \ldots \ldots + L^n)d(Tx_1, Tx_0)
$$

$$
\leq \frac{L^n}{1-L} d(Tx_1, Tx_0)
$$

Let $0 < c$ be given, choose $\rho > 0$ such that $c + \rho(0) \leq P$, where $N_\rho = \{y \in E: \|y\| < \rho\}$. Also, choose a natural number $N_1$ such that

$$
\frac{L^n}{1-L} d(Tx_1, Tx_0) \in N_\rho(0) \text{ for all } m \geq N_1.
$$

Then $\frac{L^n}{1-L} d(Tx_1, Tx_0) \ll c$, for all $m \geq N_1$. Thus,

$$
d(Tx_{2n}, Tx_{2m}) \leq \frac{L^n}{1-L} d(Tx_1, Tx_0)
$$

and

$$
\frac{L^n}{1-L} d(Tx_1, Tx_0) \ll c \text{, for all } m \geq n.
$$

Then we get $(Tx_{2n}, Tx_{2m}) \ll c$, for all $m \geq n$. Therefore, $(Tx_{2n})$ is a Cauchy sequence in $(X, d)$. Since $(X, d)$ is a complete cone metric spaces, there exist $u \in X$ such that

$$
\lim_{n \to \infty} Tx_{2n} = u \text{............... (7)}
$$

Since $T$ is subsequently convergent, $(x_{2n})$ has a convergent subsequence $(x_{2m})$ such that

$$
\lim_{m \to \infty} Tx_{2m} = v \text{............... (8)}
$$

Since $T$ is injective, then by (8), we obtain

$$
\lim_{m \to \infty} x_{2m} = Tv \text{............... (9)}
$$

By the uniqueness of the limit,

$$
u = Tv \text{........................................ (10)}
$$

Since Rand $S$ are continuous. So,

$$
\lim_{m \to \infty} Sx_{2m} = Sv \text{ and } \lim_{m \to \infty} Tx_{2m} = Rv \text{. Again ,Since } T \text{ is continuous, so,}
$$

$$
\lim_{m \to \infty} TSx_{2m} = TSv \text{ and } \lim_{m \to \infty} TRx_{2m} = TRv. \text{ Thus, if m is odd. Then,}
$$

$$
\lim_{n \to \infty} TSx_{2n+1} = TSv \text{ ..........................(11)}
$$

Choose a natural number $N_2$ such that,

$$
d(Tx_{2n+1},Tv) \ll \left[ \frac{c}{2} \left( \frac{\beta + \gamma + \delta}{1-(\beta + \delta)} \right) \right], \text{ for all } n \geq N_2.
$$

Now, consider,

$$
d(TSv, Tv) \leq d(TSv, Tx_{2n+1}) + d(Tx_{2n+1}, Tv)
$$

$$
\leq \alpha d(TSv, TSx_{2n+1}) + \beta d(Tx_{2n+1}, TRSv) + \gamma d(TSv, Tx_{2n+1})
$$

$$
+ \delta \left[ d(TSv, TRSv) + d(Tx_{2n+1}, TSx_{2n+1}) \right] + d(Tx_{2n+1}, Tv).
$$
\[
\leq \alpha d(Tv, Tx_{2n+1}) + \beta d(Tx_{2n+1}, TTSv) + \gamma d(Tv, Tx_{2n+1}) + \delta \left[ d(Tv, TTSv) + d(Tx_{2n+1}, Tx_{2n+2}) \right] + d(Tx_{2n+1}, Tv). \\
\]

So, \( d(TSv, Tv) \leq \frac{(\beta + \gamma + \delta)}{1 - (\beta + \delta)} d(Tx_{2n}, Tv) + \frac{(\alpha + \gamma)}{1 - (\beta + \delta)} d(Tv, Tx_{2n+1}) < c \), for all \( n \geq N_2 \). Therefore, \( d(TSv, Tv) < \frac{c}{i} \) for \( i \geq 1 \).

Since \( P \) is closed, \(-d(Tv, TTSv) \in P \) and \( d(TSv, Tv) < c \) for each \( c \in int P \). So now using Corollary 2.11(iii), it follows that

\[ d(Tv, TTSv) = 0 \]
which implies that \( Tv = TTSv \). As \( T \) is injective, \( v = Sv \). Thus \( v \) is the fixed point of \( S \).

Similarly, it can be established that, \( v \) is also fixed point of \( R \), that means, \( v \) is common fixed point of \( R \) and \( S \).

**Now to prove uniqueness:** Suppose that \( w \) is another common fixed point of \( R \) and \( S \), then \( Rw = w = sw \).

Now, \( d(Tv, Tw) = d(TRv, TSw) \)

\[ \leq \alpha d(Tv, TTSv) + \beta d(Tw, TRv) + \gamma d(Tv, Tw) + \delta \left[ d(Tv, TTSv) + d(Tw, TTSv) \right] \]

\[ d(Tv, Tw) \leq (\alpha + \beta + \gamma + \delta) d(Tv, Tw) \]

\[ < d(Tv, Tw) \text{ as } \alpha + \beta + \gamma + \delta < 1 \text{ a contradiction. Hence } d(Tv, Tw) = 0 \] which implies that, \( Tv = Tw \). As \( T \) is injective, \( v = w \) is the unique common fixed point of \( R \) and \( S \).

Since we have assumed that \( \{T, R\} \) and \( \{T, S\} \) are Banach pair, \( \{T, R\} \) and \( \{T, S\} \) Commutes at the fixed point of \( R \) respectively. This implies that, \( TRv = RTv \) for \( v \in F(R) \). So, \( Tv = RTv \), which gives that \( Tv \) is another fixed point of \( R \). It is also true for \( S \). By the uniqueness of fixed point of \( R, Tv = v \). Hence \( v = Tv = Rv = Sv, v \) is the unique common fixed point of \( T, R \) and \( S \) in \( X \).

4. Conclusion

In this attempt, we prove unique common fixed point results in complete cone metric spaces with two Banach pairs mapping. These results generalized and improved the concept of T-Rhoades contraction which satisfies Banach operator pair contractive conditions.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

**REFERENCES**

[1]. Banach’sSure les operations dans les ensembles abstraits et leur applications aux equations integrals, fundamental mathematicae,3(7),133-181,(1922).

© JGRMA 2017. All Rights Reserved


[48]. Al-thagafi, M.A.,and Shahzad, N., Banach operator pairs, common fixed point in variant Approximations and non expansive multimaps, Non linear Anal. 69(8), (2008), 2733-2739.
[51]. Hussain, N., Common fixed points in best approximation for Banach operator pair with
