Abstract: In this paper, I solved the problems occurred in first order ordinary linear differential equations based on RL and RC Circuits by using Laplace Transform.

Keywords: Laplace transforms, RL and RC Circuits, Differential Equations

I. INTRODUCTION
The theory of Laplace Transform is an important part of the mathematical surroundings required by engineers, physicists and mathematicians. It gives an easy and successful means for solving differential and integral equations.

The Laplace transform reduces the problem of solving differential equations to an algebraic problem. It is particularly helpful for solving problems where the mechanical or electrical driving force has discontinuities.

An integral transform called the Laplace transform defined for function of exponential order, we consider in the set A defined by

\[ \left\{ f(t) \mid \int_{0}^{\infty} e^{-st} f(t) dt < \infty \right\} \]

The satisfactory condition for the survival of the Laplace transform are that f(t) for \( t \geq 0 \) be piecewise continuous and of the exponential order otherwise Laplace transform may (or) may not exist.

The unique function \( f(t) = L^{-1} \{ f(s) \} \) is called inverse Laplace transform of \( \overline{L}(s) \) (2)

II. LAPLACE TRANSFORM OF STANDARD FUNCTIONS AND PROPERTIES

(a) \( L(e^{-at}) = \frac{1}{s + a} \) (s>a)

Proof: we have \( L(e^{-at}) = \int_{0}^{\infty} e^{-st} e^{-at} dt = \int_{0}^{\infty} e^{-(s+a)t} dt = \frac{-1}{(s+a)} [e^{-(s+a)t}]_{0}^{\infty} = \frac{1}{s + a} \)

If a=0 the \( L(1) = \frac{1}{s} \) (s>0)

Also changing the sign of ‘a’ we get \( L(e^{at}) = \frac{1}{s - a} \)
(b) \( L(\sin at) = \frac{a}{s^2 + a^2} \) and \( L(\cos at) = \frac{s}{s^2 + a^2} \) \( (s > 0) \)

**Proof:** \( L(\cos at + i \sin at) = L(e^{at}) = \frac{1}{s - ia} = \frac{s + ia}{s^2 + a^2} \)

Equating real and imaginary parts we get

\[
L(\sin at) = \frac{a}{s^2 + a^2}, \quad L(\cos at) = \frac{s}{s^2 + a^2}
\]

(c) \( L(\sinh at) = \frac{a}{s^2 - a^2} \) and \( L(\cosh at) = \frac{s}{s^2 - a^2} \) \( (s > |a|) \)

\[
L(\sinh at) = L \left( \frac{e^{at} - e^{-at}}{2} \right) = \frac{1}{2} L(e^{at}) - \frac{1}{2} L(e^{-at})
\]

**Proof:**

\[
\frac{1}{2} \left[ \frac{1}{s - a} - \frac{1}{s + a} \right], \text{Thus } \sinh(at) = \frac{a}{s^2 - a^2}
\]

Similarly, we can show that \( L(\cosh at) = \frac{s}{s^2 - a^2} \)

(d) \( L(t^n) = \frac{(n + 1)}{s^{n+1}} [(n + 1) > 0 \text{ and } s > 0] \)

**Proof:**

\[
L(t^n) = \int_0^\infty e^{-st} t^n \, dt = \int_0^\infty e^{-p} \frac{p^n}{s^n} \, dp \quad \text{where } p = st
\]

\[
= \int_0^\infty \frac{e^{-p} p^n}{s^{n+1}} \, dp = \frac{\Gamma(n + 1)}{s^{n+1}}
\]

Since \( \Gamma(n + 1) = \int_0^\infty e^{-x} x^n \, dx \) and \( \Gamma(n + 1) = n! \) if \( n \) is an integer

\[
\therefore L(t^n) = \frac{n!}{s^{n+1}}
\]

From above we have

\[
L(1) = \frac{1}{s}, \quad L(t) = \frac{1}{s^2}, \quad L(t^2) = \frac{2}{s^3}, \quad L(t^3) = \frac{6}{s^4}
\]

**Linearity Property:**

Let \( f_1(t) \) and \( f_2(t) \) be two functions defined on \([0, \infty)\) such that the Laplace transforms \( L[f_1(t)] \) and \( L[f_2(t)] \) exists. If \( k_1 \) and \( k_2 \) are two constants, then

\[
L[k_1 f_1(t) + k_2 f_2(t)] = k_1 L[f_1(t)] + k_2 L[f_2(t)]
\]

This property is valid since

\[
\int_0^\infty [k_1 f_1(t) + k_2 f_2(t)] e^{-st} \, dt = \int_0^\infty k_1 f_1(t) e^{-st} \, dt + \int_0^\infty k_2 f_2(t) e^{-st} \, dt = k_1 L[f_1(t)] + k_2 L[f_2(t)]
\]

This result can be extended to the linear combinations of more than two functions

**Laplace transforms Derivatives:**

**Theorem:** If \( f(t) \) is continuous \( \forall t \geq 0 \) and of exponential order, say \( \sigma \) and has a derivative \( f(t) \) which is piecewise continuous on every finite interval \([0, N]\) for each \( N > 0 \), then the Laplace Transform of the derivative \( f(t) \) exists for \( s > \sigma \) and

\[
L[f'(t)] = sL[f(t)] - f(0)
\]

**Proof:**
\[ L[f(t)] = \int_{0}^{\infty} e^{-st} f(t) \, dt \Rightarrow L[f'(t)] = \int_{0}^{\infty} e^{-st} f'(t) \, dt \]

Integration by parts gives
\[ L[f'(t)] = -f(0) + \int_{0}^{\infty} e^{-st} f(t) \, dt = sL[f(t)] - f(0) \]

**RL circuits:**
In a series circuit containing only a resistor and an inductor, Kirchhoff’s second law states that the sum of the voltage drop across the inductor \( L \) and voltage drop across the resistor is same as the impressed voltage \( E(t) \) on the circuit. Therefore, for the current \( i(t) \), the differential equation is
\[ L \frac{di}{dt} + Ri = E(t) \]

**Pr.1** A 12v battery is connected to a simple series circuits in which 1/2H and the resistance is 10Ω. 
Determine the current if \( i(0)=0 \).

From Kirchhoff’s second Law, we have
\[ L \frac{di}{dt} + Ri = E(t) \]

\[ \Rightarrow \frac{1}{2} \frac{di}{dt} + 10i = 12 \]

\[ \Rightarrow \frac{di}{dt} + 20i = 24 \quad (3) \]

From (1) we get,
\[ L \left( \frac{di}{dt} + 20i \right) = L(24) \]
\[ \Rightarrow L \left[ \frac{di}{dt} \right] + 20L[i(t)] = 24L[1] \]

From Laplace Transform of derivatives, we get
\[ sL[i(t)] - i(0) + 20L[i(t)] = \frac{24}{s} \]
\[ (s + 20)L[i(t)] = \frac{24}{s} \]
\[ \Rightarrow L[i(t)] = \frac{24}{s(s + 20)} \]
\[ = \frac{24}{20} \left( \frac{1}{s} - \frac{1}{s + 20} \right) \]

From (2) we get,
\[ i(t) = 6 \left[ 1 - e^{-20t} \right] \]

2. A generator having electro motive force 20cos5t volts are connected with a 10 Ω resistor and inductor 2H if the switch k is closed at t=0, obtain the differential equation for the current and determine the current at any time t.
Given \( E(t)=20\cos(5t) \) volts, \( R=10\Omega \), \( L=2\text{H} \).

The differential equation to find the current \( i(t) \) in the given current is

\[
L \frac{di}{dt} + Ri = E(t)
\]

\[
\Rightarrow 2 \frac{di}{dt} + 10i = 20 \cos(5t)
\]

\[
\Rightarrow \frac{di}{dt} + 5i(t) = 10 \cos(5t)
\]

From (1) we get

\[
L\left\{ \frac{di}{dt} + 5i(t) \right\} = L\{10 \cos(5t)\}
\]

\[
\Rightarrow L\left\{ \frac{di}{dt} \right\} + 5L\{i(t)\} = 10L\{\cos(5t)\}
\]

From Laplace Transform of derivatives, we get

\[
sL\{i(t)\} - i(0) + 5L\{i(t)\} = 10 \frac{s}{s^2 + 5^2}
\]

at \( t=0 \), current \( i(0)=0 \)

\[
(s + 5)L\{i(t)\}=10 \frac{s}{s^2 + 5^2}
\]

\[
\Rightarrow L\{i(t)\}=10 \frac{s}{(s^2 + 25)(s + 5)}
\]

\[
10 \frac{s}{(s^2 + 25)(s + 5)} = \frac{A}{s + 5} + \frac{Bs + c}{s^2 + 25}
\]

\[
10 \frac{s}{(s^2 + 25)(s + 5)} = \frac{-1}{s + 5} + \frac{s}{s^2 + 25} + \frac{5}{s^2 + 25}
\]

\[
\therefore L\{i(t)\} = -1 \frac{s}{s^2 + 25} + \frac{5}{s^2 + 25}
\]

From (2) we get

\[
i(t) = -e^{-5t} + \cos(5t) + \sin(5t)
\]

**RC- Circuits:**

The basic differential equation governing the amount of charge “q” in a simple RC Circuit consisting of a resistance R, a capacitor C, and an Electromotive force E is

\[
\frac{dq}{dt} + \frac{1}{RC}q = \frac{E(t)}{R}
\]

**Problem (3): A decaying emf \( E=200e^{-5t} \) is connected in a series with a 20Ω resistor and 0.01F capacitor. Assuming \( q=0 \) at \( t=0 \), find the charge \( q \) at any time \( t \).**

**Solution.** Given \( E(t) = 200e^{-5t}, R = 20\Omega, c = 0.01F \)

\[
\frac{dq}{dt} + \frac{1}{RC}q = \frac{E(t)}{R}
\]

\[
\frac{dq}{dt} + \frac{1}{20 \times 0.01}q = \frac{200e^{-5t}}{20}
\]

\[
\Rightarrow \frac{dq}{dt} + 5q(t) = 10e^{-6t}
\]

From (1) we get
\[ L\left\{ \frac{dq}{dt} + 5q(t) \right\} = 10L\{e^{6t}\} \]
\[ \Rightarrow L\left\{ \frac{dq}{dt} \right\} + 5L\{q(t)\} = 10L\{e^{6t}\} \]

From Laplace Transform of derivatives, we get
\[ sL\{q(t)\} - q(0) + 5L\{q(t)\} = 10 \frac{1}{s - 6} \]
\[ (s + 5)L\{q(t)\} = 10 \frac{s}{s - 6} \]
\[ \Rightarrow L\{q(t)\} = 10 \frac{1}{(s - 6)(s + 5)} \]
\[ = 10 \left[ \frac{1}{s - 6} - \frac{1}{s + 5} \right] \]
\[ L\{q(t)\} = 10 \left[ \frac{1}{s - 6} - \frac{1}{s + 5} \right]^s \]

Taking Inverse Laplace transform, we get
\[ q(t) = 10(e^{6t} - e^{-5t}) \]

IV. CONCLUSION
In the presented work, we have successfully applied Laplace Transform for solving the problems on RL and RC Electrical Circuits

V. REFERENCES