A NOTE ON SOME PROPERTIES OF A CLASS OF MODIFIED NEW BERNSTEIN TYPE OPERATORS

V. K. Gupta\textsuperscript{1}, S.K.Tiwari\textsuperscript{2} and Yogita Parmar\textsuperscript{3}

\textsuperscript{1} V. K. Gupta, Department of Mathematics, Govt. Madhav Science College Ujjain (M.P.)
Email: dr-vkg61@yahoo.co.in

\textsuperscript{2} S. K. Tiwari Department of Mathematics, School of Studies in mathematics, Vikram University, Ujjain (M.P.)
Email: Skt-tiwari75@yahoo.co.in

\textsuperscript{3} Yogita Parmar Reseach Scholar, School of Studies in mathematics, Vikram University, Ujjain (M.P.)
Email: yogitaparmar23@gmail.com

\textbf{Abstract:} In this paper we study some properties of the modified new Bernstein type operators and show that some properties of the original function such as shape preserving properties, smoothness properties, etc. which were preserved by classical Bernstein operators are also preserved by these modified operators. We also obtain an estimate on rate of convergence of these operators in terms of modul us of continuity. In this paper we use the resul tof pointwise approximation by these operators with use of Siddiqui M. A. et. al. \cite{10}.

## INTRODUCTION

For a continuous function $f$ which is defined on $[0, 1]$, the Bernstein operators $B_m$, $m \in \mathbb{N}$, are defined by

\[
(B_m f)(x) = \sum_{i=0}^{m} b_{m,i}(x)f\left(\frac{i}{m}\right), \quad x \in [0, 1]
\]  \hspace{1cm} (1)

where

\[
b_{m,i}(x) = \binom{m}{i} x^i(1-x)^{m-i}, \quad i = 0, 1, ..., m.
\]

$(B_m f)$ is a polynomial of degree atmost $m$, $m \in \mathbb{N}$ which converges to $f$ uniformly on $[0, 1]$.

In the year 2008, Deo N. et. al. \cite{2} introduced the class of new Bernstein type operators as

\[
(V_m f)(x) = \sum_{i=0}^{m} p_{m,i}(x)f\left(\frac{i}{m}\right)
\]  \hspace{1cm} (2)
where \( f \in C \left[ 0, \frac{m}{m+1} \right] \) and

\[
p_{m,i}(x) = \left( \frac{1 + m}{m} \right)^m \binom{m}{i} x^i \left( \frac{m}{m+1} - x \right)^{m-i}
\]  

(3)

In 2014, M. A. Siddqui et. al. [9] introduced a class of modified new Bernstein type operator as

\[
V_n^*(f; x) = \sum_{i=0}^{m} p_{m,i}(x) f \left( \frac{i}{m + 1} \right)
\]  

(4)

where \( f \in C \left[ 0, \frac{m}{m+1} \right] \) and \( p_{m,i}(x) \) is same as in equality (3).

It was proved in [9] that \( V_n^* f \) converge uniformly to \( f \) on \( \left[ 0, \frac{m}{m+1} \right] \). In the present paper we study some other properties of these operators.

2. Basic Result

In this section we give certain Lemmas which are necessary to prove main result. In [9] following results were proved.

Lemma 2.1 for each \( x \in \left[ 0, \frac{m}{m+1} \right], m \in \mathbb{N} \) we have,

\[
V_n^*(e_0; x) = 1
\]

\[
V_n^*(e_1; x) = x
\]

\[
V_n^*(e_2; x) = x^2 + \frac{x}{m} \left( \frac{m}{m+1} - x \right)
\]

\[
V_n^*(e_3; x) = x^3 \frac{(m-1)(m-2)}{m^2} + 3x^2 \frac{(m-1)}{m(m+1)} + \frac{x}{(m+1)^2}
\]

\[
V_n^*(e_4; x) = x^4 \frac{(m-1)(m-2)(m-3)}{m^3} + 6x^3 \frac{(m-1)(m-2)}{m^2(m+1)} + x^2 \frac{7(m-1)}{m(m+1)^2} + \frac{x}{(m+1)^3}
\]

where \( e_j(y) = y^j, \quad j = 0, 1, 2, 3, 4 \)

Lemma 2.2

\[
V_n^*(y - x; x) = 0
\]

\[
V_n^*((y - x)^2; x) = \frac{x}{m} \left( \frac{m}{m+1} - x \right)
\]
\[ V_m^*((y-x)^3;x) = \frac{2x^3}{m^2} - \frac{3x^2}{m(m+1)} + \frac{x}{(m+1)^2} \]

\[ V_m^*((y-x)^4;x) = \left( \frac{3m-6}{m^3} \right) x^4 + \frac{6x^3}{m(m+1)} + \frac{x^2}{(m+1)^2}\left( \frac{3m-7}{m} \right) + \frac{x}{(m+1)^3} \]

3. Shape preserving properties of \( V_m^* \)

In [7] certain shape preserving properties of Bernstein polynomials were established in the form of following theorem.

**Theorem 3.1** Let \( f: [0,1] \to \mathbb{R} \). Then

\[ B_m^r(f;x) = \frac{m!}{(m-r)!} \sum_{i=0}^{m-r} \binom{m-r}{i} x^i (1-x)^{m-r-i} \frac{\Delta^r_1 f}{m^r} \left( \frac{i}{m} \right), r = 0, 1, 2, ..., m \]

In particular if \( f \) is \( r \)-convex, then so is \( B_m^r(f) \).

In this section we shall establish similar properties for the operators \( (V_m^*)^r(f) \) involving forward differences of the function. The forward differences of any function \( f: [0,1] \to \mathbb{R} \) are defined as,

\[ \Delta_h f(x) = f(x + h) - f(x), \quad x, h \in [0,1], \]

\[ \Delta_h^s f(x) = \Delta_h (\Delta_h^{s-1} f(x)), \quad s \geq 2, \]

Thus \( f \) is monotonically increasing if \( \Delta_h f(x) \geq 0 \) and convex if \( \Delta_h^2 f(x) \geq 0 \), for all \( x \in [0,1] \). More generally, \( f \) is \( r \)-convex if \( \Delta_h^r f(x) \geq 0 \) for all \( x \in [0,1] \).

**Theorem 3.2** Let \( f: [0, \frac{m}{m+1}] \to \mathbb{R} \). Then

\[ (V_m^* (f;x))^r = \frac{m!}{(m-r)!} \left( \frac{1+m}{m} \right)^r \sum_{i=0}^{m-r} \binom{m-r}{i} x^i \left( \frac{m}{m+1} - x \right)^{m-r-i} \frac{\Delta^r_1 f}{m^r} \left( \frac{i}{m+1} \right) \]

where \( r = 0, 1, ..., m \). In particular if \( f \) is \( r \)-convex, then so is \( V_m^* (f) \).

**Proof.** Differentiating both side of equation (4) with respect to \( x \) we obtain.

\[ (V_m^* (f;x))' = \left( \frac{1+m}{m} \right)^r \sum_{i=0}^{m-r} \binom{m}{i} \left[ i x^{i-1} \left( \frac{m}{m+1} - x \right)^{m-i} - (m-k) x^i \left( \frac{m}{m+1} - x \right)^{m-i-1} \right] f \left( \frac{i}{m+1} \right) \]
\[
\left(1 + \frac{m}{m}\right)^m \sum_{i=1}^{m} \frac{m!}{(i-1)! (m-i)!} x^{i-1} \left(\frac{m}{m+1} - x\right)^{m-i} \frac{f(i)}{m+1} \\
- \left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-1} \frac{m!}{i! (m-i-1)!} x^i \left(\frac{m}{m+1} - x\right)^{m-i-1} \frac{f(i)}{m+1}
\]

\[
\left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-1} \frac{m(m-1)!}{i! (m-i-1)!} x^i \left(\frac{m}{m+1} - x\right)^{m-i} \frac{f(i+1)}{m+1} \\
- \left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-1} \frac{m(m-1)!}{i! (m-i-1)!} x^i \left(\frac{m}{m+1} - x\right)^{m-i} \frac{f(i)}{m+1}
\]

\[
\left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-1} \frac{m(m-1)!}{i! (m-i-1)!} x^i \left(\frac{m}{m+1} - x\right)^{m-i-1} \left[f\left(\frac{i+1}{m+1}\right) - f\left(\frac{i}{m+1}\right)\right] 
\]

\[
\frac{m!}{(m-1)!} \left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-1} \binom{m-1}{i} x^i \left(\frac{m}{m+1} - x\right)^{m-i-1} \frac{\Delta_i f\left(\frac{i}{m+1}\right)}{m+1}
\]

again differentiating,

\[
\left[V_m^* (f; x)\right]''
\]

\[
= m \left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-1} \binom{m-1}{i} \left[i x^{i-1} \left(\frac{m}{m+1} - x\right)^{m-i-1} - (m-i-1) x^i \left(\frac{m}{m+1} - x\right)^{m-i-2}\right] \frac{\Delta f\left(\frac{i}{m+1}\right)}{m+1}
\]

\[
= m \left(1 + \frac{m}{m}\right)^m \sum_{i=1}^{m-1} \frac{(m-1)!}{i! (m-i-1)!} x^{i-1} \left(\frac{m}{m+1} - x\right)^{m-i-1} \frac{\Delta f\left(\frac{i}{m+1}\right)}{m+1} \\
- m \left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-2} \frac{(m-1)!}{i! (m-i-2)!} x^i \left(\frac{m}{m+1} - x\right)^{m-i-2} \frac{\Delta f\left(\frac{i}{m+1}\right)}{m+1}
\]

\[
= m \left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-2} \frac{(m-1)(m-2)!}{i! (m-i-2)!} x^i \left(\frac{m}{m+1} - x\right)^{m-i-2} \frac{\Delta f\left(\frac{i+1}{m+1}\right)}{m+1} \\
- m \left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-2} \frac{(m-1)(m-2)!}{i! (m-i-2)!} x^i \left(\frac{m}{m+1} - x\right)^{m-i-2} \frac{\Delta f\left(\frac{i}{m+1}\right)}{m+1}
\]

\[
= m(m-1) \left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-2} \binom{m-2}{i} x^i \left(\frac{m}{m+1} - x\right)^{m-2-i} \left[\Delta f\left(\frac{i+1}{m+1}\right) - \Delta f\left(\frac{i}{m+1}\right)\right]
\]
\[
\frac{m!}{(m-2)!} \left( \frac{1 + m}{m} \right)^m \sum_{i=0}^{m-2} \binom{m-2}{i} x^i \left( \frac{m}{m+1} - x \right)^{m-2-i} \Delta^2_{m+1} f \left( \frac{i}{m+1} \right)
\]

So by induction, we have,

\[
[V^r_m(f;x)](x) = \frac{m!}{(m-r)!} \left( \frac{1 + m}{m} \right)^m \sum_{i=0}^{m-r} \binom{m-r}{i} x^i \left( \frac{m}{m+1} - x \right)^{m-r-i} \Delta^r_{m+1} f \left( \frac{i}{m+1} \right).
\]

\(r = 0, 1, \ldots, m\). Therefore, if \(f\) is \(r\)-convex, then \(\Delta^r f(x) \geq 0\) and hence \(V^r_m(f)\) is also \(r\)-convex.

Next theorem is a direct consequence of previous theorem.

**Theorem 3.3** If \(f(x)\) is non-decreasing on \([0, \frac{m}{m+1}]\) then for each \(m \in \mathbb{N}\), \(V^r_m(f;x)\) are also non-decreasing.

**Proof.** If \(f(x)\) is non-decreasing on \([0, \frac{m}{m+1}]\), then from equation (6) it follows that \(V^r_m(f;x)\)' \(\geq 0\). And hence for each \(m \in \mathbb{N}\), \(V^r_m(f;x)\) are also non-decreasing.

In [6] it was proved that Bernstein polynomials preserve starshapeness of the function. A function \(f: [0,1] \rightarrow \mathbb{R}\) is called starshaped on \([0,1]\) if,

\(f(\lambda x) \leq \lambda f(x), \forall \lambda \in [0,1], x \in [0,1]\)

If there exist \(f'(x)\) on \([0,1], f(0) = 0, f(x) \geq 0, x \in [0,1]\) then the Starshapeness can be expressed by the differential inequality, \(xf'(x) - f(x) \geq 0, \forall x \in [0,1]\). Now we show that operator \(V^r_m\) also preserve starshapeness on \([0, \frac{m}{m+1}]\), \(\forall n \in \mathbb{N}\).

**Theorem 3.4** If \(f: [0, \frac{m}{m+1}] \rightarrow \mathbb{R}\) satisfy \(f(0) = 0, f(x) \geq 0, \forall x \in [0, \frac{m}{m+1}]\) and \(f\) is starshaped on \([0, \frac{m}{m+1}]\) then \((V^r_m f)(0) = 0, (V^r_m f)(x) \geq 0, x \in [0, \frac{m}{m+1}]\) and \(V^r_m f\) is starshaped on \([0, \frac{m}{m+1}]\), \(\forall m \in \mathbb{N}\).

**Proof.** From equation (4) we have

\[
(V^r_m f)(x) = \left( \frac{m}{m+1} \right)^m \left[ \binom{m}{0} \left( \frac{m}{m+1} - x \right)^m f(0) + \sum_{i=1}^{m} \binom{m}{i} x^i \left( \frac{m}{m+1} - x \right)^{m-i} f \left( \frac{i}{m+1} \right) \right]
\]  

If \(f(0) = 0\) then clearly from equation (7), we see that \((V^r_m f)(0) = 0\). Also it is evident from definition of the operator (4) that, \(\text{iff} f(x) \geq 0, \forall x \in \left[0, \frac{m}{m+1}\right]\) then \((V^r_m f)(x) \geq 0, \forall x \in \left[0, \frac{m}{m+1}\right]\).

Now from equation (6) we have
\[ [V_m^* (f; x)]' = \left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-1} m \binom{m-1}{i} x^i \left(\frac{m}{m+1} - x\right)^{m-1-i} \left[f \left(\frac{i+1}{m+1}\right) - f \left(\frac{i}{m+1}\right)\right] \]

(8)

By definition of the operator we have

\[ \frac{V_m^* (f; x)}{x} = \left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-1} m \binom{m-1}{i} x^i \left(\frac{m}{m+1} - x\right)^{m-1-i} f \left(\frac{i}{m+1}\right) \]

\[ = \left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-1} \frac{m(m-1)!}{(i+1)! (m-1-i)!} x^i \left(\frac{m}{m+1} - x\right)^{m-1-i} f \left(\frac{i+1}{m+1}\right) \]

\[ = \left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-1} \binom{m-1}{i} \left(\frac{m}{i+1}\right) x^i \left(\frac{m}{m+1} - x\right)^{m-1-i} f \left(\frac{i+1}{m+1}\right) \]

(9)

So that from equations (8) and (9) we have

\[ [V_m^* (f; x)]' - \frac{V_m^* (f; x)}{x} = \left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-1} m \binom{m-1}{i} x^i \left(\frac{m}{m+1} - x\right)^{m-1-i} \]

\[ \times \left[f \left(\frac{i}{m+1}\right) f \left(\frac{i+1}{m+1}\right) - f \left(\frac{i}{m+1}\right)\right]\]

\[ = \left(1 + \frac{m}{m}\right)^m \sum_{i=0}^{m-1} m \binom{m-1}{i} x^i \left(\frac{m}{m+1} - x\right)^{m-1-i} A_{m,i}(f) \]

(10)

If \(f\) is starshaped then \(f(\lambda x) \leq \lambda f(x), 0 \leq \lambda \leq 1\). Then

\[ \left(\frac{i}{i+1}\right) f \left(\frac{i+1}{m+1}\right) \geq f \left(\frac{i}{i+1} \cdot \frac{i+1}{m+1}\right) = f \left(\frac{i}{i+1}\right) \]

(11)

So \(A_{m,i}(f) = \left(\frac{i}{i+1}\right) f \left(\frac{i+1}{m+1}\right) - f \left(\frac{i}{m+1}\right) \geq 0\). Hence from equation (10) we have

\[ [V_m^* (f; x)]' - \frac{V_m^* (f; x)}{x} \geq 0 \]

Hence \(V_m^* (f)\) is starshaped on \([0, \frac{m}{m+1}], \forall m \in \mathbb{N}\).

4 Some other properties of \(V_m^*\)
In this section we prove that like Bernstein operators (see[5]) operators \( V_m^* \) are also variation diminishing. Recall that a variation diminishing operator \( L_n \) has the property

\[ V[L_m f] \leq V[f], \]

where \( V[f] \) is the total variation of \( f \) as \( x \) varies across its domain.

**Theorem 4.1** For each \( m \in \mathbb{N} \) and \( f \in C \left[ 0, \frac{m}{m+1} \right] \), \( V_m^* \) is variation diminishing.

**Proof.** By definition of variation of a function and equation (6)

\[ V[V_m^* f] = \int_0^m \left| V_m^* (f; x) \right| dx \]

\[ \leq m \left( \frac{1 + m}{m} \right) \int_0^m \sum_{i=0}^{m-1} \frac{m-1}{i} x^i \left( \frac{m}{m+1} - x \right)^{m-1-i} \left| f \left( \frac{i + 1}{m + 1} \right) - f \left( \frac{i}{m + 1} \right) \right| dx \]

On substituting \( x = \frac{m}{m+1} y \), the previous inequality takes the following from

\[ V[V_m^* f] \leq m \sum_{i=0}^{m-1} \left( \frac{m-1}{i} \right) \left| f \left( \frac{i + 1}{m + 1} \right) - f \left( \frac{i}{m + 1} \right) \right| \int_0^1 y^i (1 - y)^{n-1-i} dy \]

\[ = m \sum_{i=0}^{m-1} \left( \frac{m-1}{i} \right) B(i + 1, m - i) \left| f \left( \frac{i + 1}{m + 1} \right) - f \left( \frac{i}{m + 1} \right) \right| \]

\[ = m \sum_{i=0}^{m-1} \frac{(m-1)!}{i! (m-1-i)!} \frac{i (m-1-i)!}{m!} \left| f \left( \frac{i + 1}{m + 1} \right) - f \left( \frac{i}{m + 1} \right) \right| \]

\[ = \sum_{i=0}^{m-1} \left| f \left( \frac{i + 1}{m + 1} \right) - f \left( \frac{i}{m + 1} \right) \right| \]

\[ = V[f] \]

Thus the result follows.

Now we give two more properties (see [11]) of these modified new Bernstein type operators.

**Theorem 4.2** If \( f(x) \) is a non-negative function such that \( x^{-1} f(x) \) is non-increasing on \( \left( 0, \frac{m}{m+1} \right) \) then for each \( m \geq 1 \), \( x^{-1} V_m^* f(x) \) is also non-increasing.
Proof. Suppose $f(x)$ is a non-negative function such that $x^{-1}f(x)$ is non-increasing on $\left(0, \frac{m}{m+1}\right)$. Then for $m \geq 1$.

$$
\frac{d}{dx} \left[ x^{-1}V_m^*(f; x) \right] = \frac{d}{dx} \left[ x^{-1} \sum_{i=0}^{m} P_{m,i}(x) f \left( \frac{i}{m+1} \right) \right]
$$

$$
= \left( \frac{1+m}{m} \right)^m \sum_{i=1}^{m} \binom{m}{i} f \left( \frac{i}{m+1} \right) \frac{d}{dx} \left[ x^{-1} \left( \frac{m}{m+1} - x \right)^{n-i} \right] + \frac{d}{dx} \left[ x^{-1} \left( \frac{m}{m+1} - x \right)^{m} \right] f(0)
$$

$$
= \left( \frac{1+m}{m} \right)^m \sum_{i=1}^{m} \frac{m!}{i! (m-i)!} f \left( \frac{i}{m+1} \right) \left[ (i-1)x^{i-2} \left( \frac{m}{m+1} - x \right)^{m-i} - x^{i-1} (m-i) \left( \frac{m}{m+1} - x \right)^{n-i-1} \right]

- \left( \frac{1+m}{m} \right)^m \frac{f(0)}{x^2} \left( \frac{m}{m+1} - x \right)^{m} - \left( \frac{1+m}{m} \right)^m m f(0) \frac{m}{x} \left( \frac{m}{m+1} - x \right)^{m-1}
$$

$$
= \left( \frac{1+m}{m} \right)^m \sum_{i=1}^{m-1} \binom{m-1}{i} \left( \frac{m-1}{m+1} \right) f \left( \frac{i+1}{m+1} \right) x^{i-1} \left( \frac{m}{m+1} - x \right)^{m-i-1}

- \left( \frac{1+m}{m} \right)^m \sum_{i=1}^{m-1} \binom{m-1}{i} \left( \frac{m-1}{m+1} \right) f \left( \frac{i+1}{m+1} \right) x^{i-1} \left( \frac{m}{m+1} - x \right)^{m-i-1}

- \left( \frac{1+m}{m} \right)^m \frac{f(0)}{x} \left( \frac{m}{m+1} - x \right)^{m-1} \left[ \frac{m}{x} \right] \frac{1}{x} + m
$$

$$
= \left( \frac{1+m}{m} \right)^{m-1} \sum_{i=1}^{m-1} \left[ \frac{i+1}{m+1} \right]^{-1} f \left( \frac{i+1}{m+1} \right) - \left( \frac{i}{m+1} \right)^{-1} f \left( \frac{i}{m+1} \right) \frac{m}{x} \left( \frac{m}{m+1} - x \right)^{m-1} \left[ m + \left( \frac{m}{m+1} \right) \frac{1}{x} \right]
$$

This is non-positive by assumption. Hence $x^{-1}V_m^*(f; x)$ is non-increasing.

Recall that a function $\omega(y)$ on $[0, 1]$ is called a modulus of continuity if $\omega(y)$ is continuous, non-decreasing, semi-additive, and $\lim_{y \to 0^+} \omega(y) = \omega(0) = 0$.

**Theorem 4.3** If $\omega(y)$ is a modulus of continuity then for each $m \geq 1, V_m^*(\omega; y)$ is also a modulus of continuity.

**Proof** For any modulus of continuity $\omega(y)$ and $m \geq 1$ we see that
\[
\lim_{y \to 0} V_m^*(\omega; y) = V_m^*(\omega; 0) = \omega(0) = 0
\]

and \( V_m^*(\omega; y) \) is continuous and non-decreasing. Let \( x_1 \leq x_2 \) be any two points in \([0, \frac{m}{m+1}]\) where \( \frac{m}{m+1} \geq \max\{x_1, x_2\}. \) Then following \([1]\) we have from equation (4)

\[
V_m^*(f; x_2) = \sum_{j=0}^{m} \left( \frac{1 + m}{m} \right) \left( \frac{m}{m+1} - x_2 \right)^m f \left( \frac{j}{m+1} \right)
\]

\[
= \sum_{j=0}^{m} \left( \frac{1 + m}{m} \right) \left( \frac{m}{m+1} - x_2 \right)^m f \left( \frac{j}{m+1} \right) \left\{ \sum_{i=0}^{j} (j)_x (x_2 - x_1)^{j-i} \right\}
\]

\[
= \sum_{j=0}^{m} \sum_{i=0}^{j} \left( \frac{1 + m}{m} \right) \frac{m!}{i! (m-j)! (j-i)!} \left( \frac{m}{m+1} - x_2 \right)^m f \left( \frac{j}{m+1} \right)
\]

On inverting the order of summation and writing \( i + l = j, \) then

\[
V_m^*(f; x_2) = \sum_{i=0}^{m} \sum_{l=0}^{m-i} \left( \frac{1 + m}{m} \right) \left( \frac{m}{m+1} - x_2 \right)^m m! (x_2 - x_1)^{i-l} \left( \frac{m}{m+1} - x_2 \right)^m f \left( \frac{i + l}{m+1} \right) \] (12)

again from (4)

\[
V_m^*(f; x_1) = \sum_{i=0}^{m} \left( \frac{1 + m}{m} \right) \left( \frac{m}{m+1} - x_2 \right)^m f \left( \frac{i}{m+1} \right)
\]

\[
= \sum_{i=0}^{m} \left( \frac{1 + m}{m} \right) \left( \frac{m}{m+1} - x_2 \right)^m \left( x_2 - x_1 \right)^l \left( \frac{m}{m+1} - x_2 \right)^m f \left( \frac{i}{m+1} \right)
\]

\[
= \sum_{i=0}^{m} \sum_{l=0}^{m-i} \left( \frac{1 + m}{m} \right) \frac{m!}{i! (m-i-l)!} \left( x_2 - x_1 \right)^l \left( \frac{m}{m+1} - x_2 \right)^m f \left( \frac{i}{m+1} \right) \] (13)

From (12) and (13) using semi-additive of \( \omega(y) \) we have, for \( 0 \leq y_1 < y_2 \leq \frac{m}{m+1} \)

\( y_1 + y_2 \leq \frac{m}{m+1} \)

\[
V_m^*(\omega; y_2) - V_m^*(\omega; y_1)
\]
\[
= \sum_{i=0}^{m-1} \sum_{l=0}^{m-i} \left( \frac{1 + m}{m} \right)^m \frac{m!}{i! (m - i)!} \left( \frac{m - y_2 - y_1^l}{m + 1} \right)^{m-i} \left( \frac{\omega}{m + 1} \right) + \omega \left( \frac{i + l}{m + 1} \right)
\]

\[
\leq m \sum_{i=0}^{m-1} \sum_{l=0}^{m-i} \left( \frac{1 + m}{m} \right)^m \frac{m!}{i! (m - i)!} \left( \frac{m - y_2 - y_1^l}{m + 1} \right)^{m-i} \left( \frac{\omega}{m + 1} \right)
\]

\[
= \sum_{l=0}^{m} \left( \frac{1 + m}{m} \right)^m \frac{(y_2 - y_1^l)!}{l! (m - l)!} \left( \frac{\omega}{m + 1} \right) \left[ \sum_{i=0}^{m-l} \left( \frac{m - l}{i} \right) y_1^l \left( \frac{m - y_2}{m + 1} \right)^{m-l-i} \right]
\]

\[
= \sum_{l=0}^{m} \left( \frac{1 + m}{m} \right)^m \left( \frac{m}{l} \right) (y_2 - y_1^l) \left( \frac{m - y_2 + y_1}{m + 1} \right)^{m-l} \omega \left( \frac{l}{m + 1} \right)
\]

= \mathcal{V}_m^*(\omega; y_2 - y_1)

This shows that \(\mathcal{V}_m^*(\omega; y)\) is semi-additive and hence \(\mathcal{V}_m^*(\omega; y)\) is a modulus of continuity.

5. Rate of convergence of \(\mathcal{V}_m^*(f)\)

In this section we compute rates of convergence of the operators \(\mathcal{V}_m^* f\) by the means of first and second modulus of continuities. Let \(f \in C[0, a]\). The modulus of continuity of \(f\) denoted by \(\omega(f, \delta)\) is defined to be

\[
\omega(f, \delta) = \sup_{|s-x|<\delta, x \in [0, a]} |f(s) - f(x)|
\]

The modulus of continuity of the function \(f\) in \(C[0, a]\) gives the maximum oscillation of \(f\) in any interval of length not exceeding \(\delta > 0\). It is well known that a necessary and sufficient condition for a function \(f\) to be in \(C[0, a]\) is

\[
\lim_{\delta \to 0} \omega(f, \delta) = 0.
\]

It is also well-known that for any \(\delta > 0\) we have

\[
|f(s) - f(x)| \leq \omega(f, \delta) \left( \frac{|s-x|}{\delta} + 1 \right)
\]  

(14)

**Theorem 5.1** Let \(f \in C \left[0, \frac{m}{m+1}\right], \mathcal{V}_m^*(f; x)\) be given by (4), then

\[
\|\mathcal{V}_m^*(f; \cdot) - f\| \leq \frac{3}{2} \omega(f, \delta_m)
\]

where \(\delta_m = m^{-\frac{1}{2}}\)
**Proof.** For the proof we use similar technique of Popoviciu [8]. Let \( \omega(f, \delta) \) denote the modulus of continuity of a function \( f \), then we can write for any \( \delta > 0 \)

\[
|f(y) - f(x)| \leq \omega(f, \delta) \left( \frac{|y - x|}{\delta} + 1 \right)
\]  

(15)

Now using linearity and positive of \( V_m^*(f; x) \) and (15) for \( m \in N \) and \( x \in \left[0, \frac{m}{m+1}\right] \), we have

\[
|V_m^*(f; x) - f(x)| \leq V_m^*(|f(y) - f(x)|; x)
\]

\[
\leq \omega(f, \delta) \left( \frac{V_m^* (|y - x|; x)}{\delta} + 1 \right)
\]

\[
\leq \omega(f, \delta) \left( \frac{(V_m^* ((y - x)^2; x))^{1/2}}{\delta} + 1 \right)
\]

\[
= \omega(f, \delta) \left( \frac{1}{\delta} \sqrt{\frac{x}{m}} \left( \frac{m}{m+1} - x \right) + 1 \right)
\]

We see that the maximum value of \( \frac{x}{m} \left( \frac{m}{m+1} - x \right) \) in the interval \( \left[0, \frac{m}{m+1}\right] \) is \( \frac{m}{4(m+1)^2} \), so we have

\[
\|V_m^*(f; \cdot) - f\| \leq \omega(f, \delta) \left( \frac{1}{\delta} \sqrt{m} + 1 \right)
\]

\[
= \omega(f, \delta) \left( \frac{m}{2\delta m^{1/2}(m+1)} + 1 \right)
\]

\[
\leq \omega(f, \delta) \left( \frac{1}{2\delta m^{1/2}} + 1 \right)
\]

If we choose \( \delta = \delta_m = m^{-1/2} \), then

\[
\|V_m^*(f; \cdot) - f\| \leq \frac{3}{2} \omega(f, \delta_m)
\]

Ditzain in [3] gave a direct estimate for the Bernstein polynomials using Ditzian-Totik moduli of smoothness. Motivated from it we give a direct theorem for the operators in equation (4). For this we give some notations. Let \( C \left[0, \frac{m}{m+1}\right] \) be the set of continuous and bounded functions on \( \left[0, \frac{m}{m+1}\right] \). Ditzain-Totik moduli of smoothness is given by

\[
\omega_{\varphi, \lambda}^2(f, y) = \sup_{u, \lambda \leq y} \sup_{x \in \left[0, \frac{m}{m+1}\right]} \left| f \left( x - h \varphi^\lambda(x) \right) - 2f(x) + f(x + h \varphi^\lambda(x)) \right|
\]

(16)
where $\varphi(x)^2 = x \left( \frac{m}{m+1} - x \right)$ and K-functional

$$K_{\varphi, \lambda}(f, y^2) = \inf \left\{ \| f - g \|_{C[0, \frac{m}{m+1}]} + y^2 \| \varphi^{2\lambda} g'' \|_{C[0, \frac{m}{m+1}]} \right\}$$

(17)

where infimum is taken on functions satisfying $g, g' \in A_{C_{loc}}$.

It is well known (see[4], Theorem 3.1.2) that $\omega_{\varphi, \lambda}^2(f, y)$ is equivalent to $K_{\varphi, \lambda}(f, y^2)$. That means there exists $C > 0$ such that

$$C^{-1}K_{\varphi, \lambda}(f, y^2) \leq \omega_{\varphi, \lambda}^2(f, y) \leq CK_{\varphi, \lambda}(f, y^2)$$

(18)

**Theorem 5.2** If $f \in C \left[ 0, \frac{m}{m+1} \right], 0 \leq \lambda \leq 1$, then we have

$$|V_m^*(f; x) - f(x)| \leq Cm^{-1/2} \varphi(x)^{1-\lambda}$$

(19)

where $\varphi(x)^2 = x \left( \frac{m}{m+1} - x \right)$

**Proof.** Using (17) and (18), we may choose $g = g_{m, x, \lambda}$ for a fixed $x$ and $\lambda$ such that

$$\| f - g \| \leq A\omega_{\varphi, \lambda}^2(f, m^{-1/2} \varphi(x)^{1-\lambda})$$

(20)

$$m^{-1} \varphi(x)^{2-2\lambda} \| \varphi^{2\lambda} g'' \| \leq B\omega_{\varphi, \lambda}^2(f, m^{-1/2} \varphi(x)^{1-\lambda})$$

(21)

So by linearity of $V_m^*(f)$ and using (20) and (21) we have

$$|V_m^*(f; x) - f(x)| \leq |V_m^*(f - g; x)| + |f(x) - g(x)| + |V_m^*(g; x) - g(x)|$$

$$\leq 2\| f - g \| + |V_m^*(g; x) - g(x)|$$

$$\leq 2A\omega_{\varphi, \lambda}^2(f, m^{-1/2} \varphi(x)^{1-\lambda}) + |V_m^*(g; x) - g(x)|$$

(22)

Using Tylor expansion with integral remainder

$$f(y) = f(x) + f'(x)(y - x) + \int_x^y (y - u)f''(u)\,du$$

(23)

From ([4], p141) for $y < u < x$ we have

$$\frac{|y - u|}{\varphi(u)^{2\lambda}} = \frac{|y - x|}{\varphi(x)^{2\lambda}}$$

(24)
Hence from lemma 2.1 & 2.2 and (23), (24) we get

\[ |V_m^*(g; x) - g(x)| \leq |g'(x)||V_m^*(y - x; x) + \left| V_m^* \left( \int_x^y (t - u)g''(u)du ; x \right) \right| \]

\[ \leq V_m^* \left( |y - x| \right) \left| \int_x^y \varphi(u)^{2\lambda} g''(u)du \right| ; x \]

\[ \leq \| \varphi^{2\lambda} g'' \| V_m^* \left( \frac{(y - x)^2}{\varphi(x)^{2\lambda}} ; x \right) \]

\[ = \| \varphi^{2\lambda} g'' \| \frac{m^{-1} \varphi(x)^2}{\varphi(x)^{2\lambda}} \]

\[ = m^{-1} \varphi(x)^{2-2\lambda} \| \varphi^{2\lambda} g'' \| \]

\[ \leq \omega_2 \left( f, m^{-1/2} \varphi(x)^{1-\lambda} \right) \]  \hspace{1cm} (25)

From (22) and (25) we get the desired result.

REFERENCES


[8] Popovicin T., Sur l’approximatin des fonctions convexes d’ordre sup’erieur, Mathematica (Cluj), 10(1934), 49-54.
