RICCI SOLITONS IN KENMOTSU MANIFOLDS

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Abstract: The Ricci Soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold. In this paper we will find that the condition for Ricci Soliton in Kenmotsu manifolds to be Shrinking, Steady and Expanding.

Keywords: Bocnher Curvature tensor, Ricci tensor, Curvature tensor.

INTRODUCTION:

All calculation considered in this paper are simple and connected. One of a Ricci Soliton is a triple \(( g, v, \lambda )\) with \( g \) a Riemannian metric, \( v \) a vector field and \( \lambda \) is a real scalar such that

\[
\mathcal{L}_v g + 2S + 2\lambda g = 0
\]

Where \( S \) is a Ricci tensor of Riemannian manifold \(( M, g )\) and \( \mathcal{L}_v \) denotes the Lie derivative operator along the vector field \( V \). The Ricci soliton is said to be Shrinking, Steady and Expanding according as \( \lambda \) is negative, zero and positive respectively [3].

In 1972 Kenmotsu [9] studied a class of contact Riemannian manifold satisfying some special conditions and this manifold is known as Kenmotsu manifold. Kenmotsu proved that a locally kenmotsu manifold is a warped product \( I \times j^N \) of an interval \( I \) and a Kaehler manifold \( N \) with warping function \( f(t) = Se^t \), where \( S \) is a non-zero constant.

PRELIMINARIES

An \( n \)-dimensional differential manifold \( M \) is said to be an almost contact metric manifold [3]. If it admits an almost contact metric structure \(( \phi, \xi, \eta, g)\) consisting of a tensor field \( \phi \) of type \(( 1,1)\) a vector field \( \xi \), a 1-form \( \eta \), and a Riemannian metric \( g \) compatible with \(( \phi, \xi, \eta, g)\) satisfying
\( \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \) \tag{1}

\( \eta \circ \phi = 0, \phi \xi = 0 \) \tag{2}

\( g(\phi X, \phi Y) = g(X, Y) - \eta(X) \eta(Y), \quad g(X, \xi) = \eta(X), \) \tag{3}

for all vector fields \( X, Y \) on \( M. \)

An almost contact metric manifold \( M(\phi, \xi, \eta, g) \) is said to be Kenmotsu manifold if

\[ (\nabla_X \phi) Y = g(\phi X, Y) \xi - \eta(Y) \phi X. \] \tag{4}

From equation (4), we have

\[ \nabla_X \xi = X - \eta(X), \] \tag{5}

where \( \nabla \) denotes the Riemannian connection of \( g. \) In an \( n \)-dimensional Kenmotsu manifold, we have

\[ \eta(R(X, Y)Z) = g(X, Z) \eta(Y) - g(Y, Z) \eta(X), \] \tag{6}

\[ R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \] \tag{7}

\[ R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \] \tag{8}

\[ R(\xi, X)\xi = X - \eta(X)\xi, \] \tag{9}

where \( R \) is the Riemannian curvature tensor.

Let \( (g, V, \lambda) \) be a Ricci soliton is an \( n \)-dimensional Kenmotsu manifold \( M. \)

From equation (5), we have

\[ (L_\xi g)(X, Y) = 2[g(X, Y) - \eta(X)\eta(Y)], \] \tag{10}

From equations (1) and (10), we have

\[ S(X, Y) = -(\lambda + 1)g(X, Y) + \eta(X)\eta(Y), \] \tag{11}

\[ S(\phi X, \phi Y) = S(X, Y) + \lambda \eta(X)\eta(Y) \] \tag{12}
\[QX = -(\lambda + 1)X + \eta(X)\xi,\]  
\[S(X, \xi) = -\lambda \eta(X),\]  
\[r = -\lambda n - (n - 1),\]

where S is the Ricci tensor, Q is the Ricci operator and r is the scalar curvature on M.

**Theorem 1.1:** A Ricci soliton in a Kenmotsu Manifold satisfying \(R(\xi, X).B = 0\) is expanding.

**Proof:** Suppose \(R(\xi, X).B(Y, Z)W = 0\) then we have


Now using equation (8) in above equation, we get

\[\eta(B(Y, Z)W - g(X, B(Y, Z)W)\xi - B(\eta(Y)X - g(X, Y)\xi, Z)W - B(Y, \eta(Z)X - g(X, Z)\xi)W - B(Y, Z)\eta(W)X - g(X, W)\xi) = 0\]

\[\Rightarrow \eta(B(Y, Z)W)X - g(X, B(Y, Z)W)\xi - (1 - \lambda + \frac{4}{n+3})g(X, W)\eta(Z) - g(Z, W)\eta(X)\eta(Y) - (1 - \lambda + \frac{4}{n+3})g(X, W)\eta(Z) - g(Z, W)\eta(X)\eta(Y) = 0.\]

Taking an inner product of above equation with \(\xi\), we get

\[\eta(B(Y, Z)W)X - g(X, B(Y, Z)W) - g(X, B(Y, Z)W)\xi - B(\eta(Y)X - g(X, Y)\xi, Z)W - B(Y, \eta(Z)X - g(X, Z)\xi)W - B(Y, Z)\eta(W)X - g(X, W)\xi) = 0.\]

\[\Rightarrow \eta(B(Y, Z)W)X - g(X, B(Y, Z)W)\xi - B(\eta(Y)X - g(X, Y)\xi, Z)W - B(Y, \eta(Z)X - g(X, Z)\xi)W - B(Y, Z)\eta(W)X - g(X, W)\xi) = 0.\]

\[\Rightarrow (1 - \lambda + \frac{4}{n+3})g(X, W)\eta(Y) - g(Z, W)\eta(X)\eta(Y) = 0.\]

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\[\Rightarrow (1 - \lambda + \frac{4}{n+3})g(X, Y)\eta(Y) - g(X, W)\eta(Y) = 0.\]
\[ Z) \ g(X, \phi W)] - \frac{D}{n+3} [\eta(Z)\eta(X)g(Y, W) - \eta(Z)\eta(W)g(X, Y) + \eta(Y)\eta(W)g(X, Z) - \eta(Y)\eta(X)g(Z, W)] + \frac{D-4}{n+3} \\
[g(Y, W) g(Z, X) - g(Z, W) g(X, Y) = 0.
\]

Taking \( X = Y = e \) in above equation and summing over \( i, 1 \leq i \leq n \), we get

\[
[1 - \lambda \frac{n+3}{n+3} + \frac{4}{n+3}] [\ g(Z, W) - n g(Z, W)] - S(Z, W) - \frac{1}{n+3} [S(Z, W) - n S(Z, W) - r g(Z, W) + S(Z, W) - S(\phi Z, \phi W) - S(\phi Z, \phi W) - S(\phi W, \phi Z) + r \eta(Z)\eta(W) + \lambda \eta(Z)\eta(W) + S(Z, W) + \lambda \eta(Z)\eta(W) + D+1 \frac{n-1}{n+3} [-g(\phi Z, \phi W) - 2g(\phi Z, \phi W)] - \frac{D}{n+3} [\eta(Z)\eta(W) - n \eta(Z)\eta(W) + \eta(Z)\eta(W) - g(Z, W)] + \frac{D-4}{n+3} [g(Z, W)] - n g(Z, W)] = 0,
\]

\[
\Rightarrow [1 - \lambda \frac{n+3}{n+3} + \frac{4}{n+3}] [1 - (n) g(Z, W) - S(Z, W) - \frac{1}{n+3} [S(Z, W) - n S(Z, W) - r g(Z, W) + (r - 4\lambda) \eta(Z)\eta(W)] + \frac{D+1}{n+3} [-3g(Z, W) + 3 \eta(Z)\eta(W)] - \frac{D}{n+3} [2n \eta(Z)\eta(W) - g(Z, W)] = 0,
\]

\[
\Rightarrow [1 - \lambda \frac{n+3}{n+3} + \frac{4}{n+3}] [1 - (n) g(Z, W) - S(Z, W) - \frac{1}{n+3} [(n+3)(\lambda+1)g(Z, W)-(n+3) \eta(Z)\eta(W) - rg(Z, W) + (r-4\lambda) \\
\eta(Z)\eta(W)] + \frac{D+1}{n+3} [-3g(Z, W) + 3 \eta(Z)\eta(W)] - \frac{D}{n+3} [2n \eta(Z)\eta(W) - g(Z, W)] = 0,
\]

\[
\Rightarrow [1 - \lambda \frac{n+3}{n+3} + \frac{4}{n+3}] [1 - (n+1) - (\lambda+1) + \frac{r}{n+3} - 3\frac{D+1}{n+3} + \frac{D}{n+3}] [g(Z, W) + [\frac{1}{n+3} (r - 4 \lambda - 3n) + 3\frac{D+1}{n+3} - \frac{2-n}{n+3}] \\
\eta(Z)\eta(W) = S(Z, W)
\]

\[
(n+3) S(Z, W) = [-n (n+3) - 4 \lambda + (n-3)D + r -3n + 3]g(Z, W) + [-r + 4 \lambda + 4n + (n+1) D] \eta(Z)\eta(W).
\]

Putting \( Z = W = \xi \) in above equation and using equations (11) and (15), we get

\[
\lambda = (n-1).
\]

This shows that \( \lambda \) is positive that is, the Ricci soliton in Kenmotsu manifold satisfying \( R(\xi, X) \cdot B = 0 \) is expanding.

**REFERENCES**


