C-Class Function on New Contractive Conditions of Integral Type on Complete S-Metric Spaces

Arslan Hojat Ansari\textsuperscript{1}, D. Dhamodharan\textsuperscript{2}, Yumnam Rohen\textsuperscript{3}, R. Krishnakumar\textsuperscript{4}

\textsuperscript{1}Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran
analisisamirmath2@gmail.com\textsuperscript{1}

\textsuperscript{2}Department of Mathematics, Jamal Mohamed College (Autonomous), Tiruchirappalli-620020, India.
dharanraj28@yahoo.co.in\textsuperscript{2}

\textsuperscript{3} Department of Mathematics, National Institute of Technology, Manipur, Imphal-795004, India
ymnehor2008@yahoo.com\textsuperscript{3}

\textsuperscript{4}Department of Mathematics, Urumu Dhanalakshmi College, Tiruchirappalli-620019, India
srksacet@yahoo.co.in\textsuperscript{4}

Abstract: In this paper, we generalised the concept of a new contractive conditions of integral type on complete S-metric spaces via C-class function.

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INTRODUCTION AND MATHEMATICAL PRELIMINARIES


In this paper we discuss generalised result on C-class function on new contractive conditions of integral type on complete S-metric spaces.

Definition 1.1 Let $X \neq \emptyset$ be any set and $S : X \times X \times X \rightarrow [0, \infty)$ be a function satisfying the following conditions for all $u, v, z, a \in X$.

1. $S(u, v, z) \geq 0$
2. \( S(u, v, z) = 0 \) if and only if \( u = v = z \).

3. \( S(u, v, z) \leq S(u, u, a) + S(v, v, a) + S(z, z, a) \)

Then the function \( S \) is called an \( S \)-metric on \( X \) and the pair \( (X, S) \) is called an \( S \)-metric space simply SMS.

**Example 1.2** [3] Let \( X \) be a non empty set, \( d \) is ordinary metric space on \( X \), then \( S(x, y, z) = d(x, z) + d(y, z) \) is an \( S \)-metric on \( X \).

**Lemma 1.3** Let \( (X, S) \) be an \( S \)-metric space. Then we have \( S(u, u, v) = S(v, v, u) \)

**Definition 1.4** Let \( (X, S) \) be an \( S \)-metric space.

1. A sequence \( \{u_n\} \) in \( X \) converges to \( u \) if and only if \( S(u_n, u_n, u) \rightarrow 0 \) as \( n \rightarrow \infty \). That is, there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0, S(u_n, u_n, u) < \varepsilon \). We denote this by \( \lim_{n \rightarrow \infty} u_n = u \) or \( \lim_{n \rightarrow \infty} S(u_n, u_n, u) = 0 \).

2. A sequence \( \{u_n\} \) in \( X \) is called a Cauchy sequence if \( S(u_n, u_n, u_m) \rightarrow 0 \) as \( n, m \rightarrow \infty \). That is, there exists \( n_0 \in \mathbb{N} \) such that for all \( n, m \geq n_0, S(u_n, u_n, u_m) < \varepsilon \).

3. The \( S \)-metric space \( (X, S) \) is called complete if every Cauchy sequence is convergent.

In the following lemma we see the relationship between a metric and an \( S \)-metric.

**Lemma 1.5** Let \( (X, d) \) be a metric space. Then the following properties are satisfied:

1. \( S_d(u, v, z) = d(u, z) + d(v, z) \) for all \( u, v, z \in X \) is an \( S \)-metric on \( X \).

2. \( u_n \rightarrow u \) in \( \{X, d\} \) if and only if \( u_n \rightarrow u \) in \( (X, S_d) \).

3. \( \{u_n\} \) is Cauchy in \( \{X, d\} \) if and only if \( \{u_n\} \) is Cauchy in \( (X, S_d) \).

4. \( \{X, d\} \) is complete if and only if \( (X, S_d) \) is complete.
Definition 1.6 [1] A mapping \( F : [0, \infty)^2 \to [0, \infty) \) is called a \( C \)-class function if it is continuous and satisfies following axioms:

1. \( F(s,t) \leq s \);
2. \( F(s,t) = s \) implies that either \( s = 0 \) or \( t = 0 \); for all \( s,t \in [0,\infty) \).

Note for some \( F \) we have that \( F(0,0) = 0 \).
We denote \( C \)-class functions as \( C \).

Example 1.7 [1] The following functions \( F : [0, \infty)^2 \to \mathbb{R} \) are elements of \( C \), for all \( s,t \in [0,\infty) \):

1. \( F(s,t) = s - t, \) \( F(s,t) = s \Rightarrow t = 0; \)
2. \( F(s,t) = ms, \) \( 0 < m < 1, \) \( F(s,t) = s \Rightarrow s = 0; \)
3. \( F(s,t) = \frac{s}{(1+t)^r}; \) \( r \in (0,\infty), \) \( F(s,t) = s \Rightarrow s = 0 \) or \( t = 0; \)
4. \( F(s,t) = \log(t+a^r)/(1+t), \) \( a > 1, \) \( F(s,t) = s \Rightarrow s = 0 \) or \( t = 0; \)
5. \( F(s,t) = \ln(1+a^s)/2, \) \( a > e, \) \( F(s,1) = s \Rightarrow s = 0; \)
6. \( F(s,t) = (s+l)^(1/(1+l^r)) - l, \) \( l > 1, r \in (0,\infty), \) \( F(s,t) = s \Rightarrow t = 0; \)
7. \( F(s,t) = s \log_{e^a} a, \) \( a > 1, \) \( F(s,t) = s \Rightarrow s = 0 \) or \( t = 0; \)
8. \( F(s,t) = s - \left(\frac{1+s}{2+s}\right)^{2+t}, \) \( F(s,t) = s \Rightarrow t = 0; \)
9. \( F(s,t) = s \beta(s), \) \( \beta : [0, \infty) \to (0,1), \) and is continuous, \( F(s,t) = s \Rightarrow s = 0; \)
10. \( F(s,t) = s - \frac{t}{k+t}, F(s,t) = s \Rightarrow t = 0; \)
11. \( F(s,t) = s - \varphi(s), F(s,t) = s \Rightarrow s = 0, \) here \( \varphi : [0, \infty) \to [0, \infty) \) is a continuous function such that \( \varphi(t) = 0 \Leftrightarrow t = 0; \)
12. \( F(s,t) = sh(s,t), F(s,t) = s \Rightarrow s = 0, \) here \( h : [0, \infty) \times [0, \infty) \to [0, \infty) \) is a continuous function such that \( h(t,s) < 1 \) for all \( t,s > 0; \)
13. \( F(s,t) = s - \left(\frac{2+t}{1+t}\right)t, F(s,t) = s \Rightarrow t = 0. \)
14. \( F(s,t) = \sqrt[n]{\ln(1+s^n)}, F(s,t) = s \Rightarrow s = 0. \)
15. \( F(s,t) = \phi(s), F(s,t) = s \Rightarrow s = 0, \) here \( \phi : [0, \infty) \to [0, \infty) \) is a upper semi continuous function such that \( \phi(0) = 0 \), and \( \phi(t) < t \) for \( t > 0, \)
16. \( F(s,t) = \frac{s}{(1+s)^r}, r \in (0,\infty), F(s,t) = s. \Rightarrow s = 0; \)

Definition 1.8 [1] A function \( \psi : [0, \infty) \to [0, \infty) \) is called an altering distance function if the following properties are satisfied:

\( \psi \) is non-decreasing and continuous,
\( \psi(t) = 0 \) if and only if \( t = 0. \)
Definition 1.9 [1] An ultra altering distance function is a continuous, nondecreasing mapping \( \varphi: P \to P \) such that \( \varphi(t) > 0, \ t \in [0, \infty) \) and \( \varphi(0) \geq 0. \)

We denote this set with \( \Phi_u \)

Definition 1.10 [13] The set \( \{ a = x_0, x_1, x_2, \ldots, x_n = b \} \) is called a partition for \( [a, b] \) if and only if the sets \( \{ x_{i-1}, x_i \}_{i=1}^n \) are pairwise disjoint and \( [a, b] = \bigcup_{i=1}^{n} [x_{i-1}, x_i) \cup \{b\} \)

Definition 1.11 [13] The function \( \zeta: [0, \infty) \to [0, \infty) \) is called subadditive integrable function if and only if for all \( a, b \in P, \)

\[
\int_{0}^{a+b} \zeta(t) \, dt \leq \int_{0}^{a} \zeta(t) \, dt + \int_{0}^{b} \zeta(t) \, dt
\]

Example 1.12 Let \( E = X = R, d(x, y) = |x - y|, P = (0, \infty), \) and \( \zeta(t) = \frac{1}{(t+1)} \) for all \( t > 0. \) Then for all \( a, b \in P, \)

\[
\int_{0}^{a+b} \frac{dt}{(t+1)} = \ln(a+b+1), \int_{0}^{a} \frac{dt}{(t+1)} = \ln(a+1), \int_{0}^{b} \frac{dt}{(t+1)} = \ln(b+1)
\]

Since \( ab \geq 0, \) then

\[
a + b + 1 \leq a + b + 1 + ab = (a + 1)(b + 1).
\]

Therefore

\[
\ln(a+b+1) \leq \ln(a+1) \leq \ln(b+1)
\]

This shows that \( \zeta \) is an example of subadditive integrable function.

Theorem 1.13 [4] Let \( (X, S) \) be a complete \( S \)-metric space, \( h \in (0,1), \) the function \( \zeta: [0, \infty) \to [0, \infty) \) be defined as for each \( \varepsilon > 0, \int_{0}^{\varepsilon} \zeta(t) \, dt > 0 \) and \( T: X \to X \) be a self-mapping of \( X \) such that

\[
\int_{0}^{S(Tu,Tv,Tw)} \zeta(t) \, dt \leq h \int_{0}^{S(u,v)} \zeta(t) \, dt
\]

for all \( u, v \in X. \) Then \( T \) has a unique fixed point \( w \in X \) and we have \( \lim_{n \to \infty} T^n u = w, \) for each \( u \in X. \)

Theorem 1.14 [4] Let \( (X, S) \) be a complete \( S \)-metric space, the function
\( \zeta : [0, \infty) \to [0, \infty) \) be defined as for each \( \varepsilon > 0, \int_0^\varepsilon \zeta(t) \, dt > 0 \) and \( T : X \to X \) be a self-mapping of \( X \) such that

\[
\int_0^{\zeta(TuTv)} \zeta(t) \, dt \leq h_1 \int_0^{\zeta(uTv)} \zeta(t) \, dt + h_2 \int_0^{\zeta(uTv)} \zeta(t) \, dt + h_3 \int_0^{\zeta(uTv)} \zeta(t) \, dt + h_4 \int_0^{\zeta(uTv)} \zeta(t) \, dt + h_5 \int_0^{\zeta(uTv)} \zeta(t) \, dt.
\]

for all \( u, v \in X \) with non-negative real numbers \( h_i (i \in \{1, 2, 3, 4\}) \) satisfying \( \max \{h_1 + 3h_3, 2h_4, h_1 + h_2 + h_3\} < 1 \). Then \( T \) has a unique fixed point \( w \in X \) and we have \( \lim_{n \to \infty} T^n u = w \), for each \( u \in X \).

**MAIN RESULT**

**Theorem 2.1** Let \( (X, S) \) be a complete \( S \)-metric space \( \psi : [0, \infty) \to [0, \infty) \) is an altering distance function, \( \varphi \in \Phi_u \) and \( F \in C \), the function \( \zeta : P \to P \) be defined as for each \( \varepsilon > 0, \int_0^\varepsilon \zeta(t) \, dt > 0 \) and \( T : X \to X \) be a self-mapping of \( X \) such that

\[
\psi(\int_0^{S(uTv)} \zeta(t) \, dt) \leq F(\psi(\int_0^{S(uTv)} \zeta(t) \, dt), \varphi(\int_0^{S(uTv)} \zeta(t) \, dt)).
\]

for all \( u, v \in X \), Then \( T \) has a unique fixed point \( w \in X \) and we have \( \lim_{n \to \infty} T^n u = w \), for each \( u \in X \).

**Proof.** Let \( u_0 \in X \) and the sequence \( \{u_n\} \) be defined as \( T^n u_0 = u_n \). Suppose that \( u_n \neq u_{n+1} \) for all \( n \). Using the inequality (2.1), we obtain

\[
\psi(\int_0^{S(uTv)} \zeta(t) \, dt) \leq F(\psi(\int_0^{S(uTv)} \zeta(t) \, dt), \varphi(\int_0^{S(uTv)} \zeta(t) \, dt)) \leq \psi(\int_0^{\zeta(t)} \zeta(t) \, dt).
\]

so

\[
\int_0^{S(uTv)} \zeta(t) \, dt \leq \int_0^{\zeta(t)} \zeta(t) \, dt
\]

Since \( \int_0^{\zeta(t)} \zeta(t) \, dt > 0 \), there exists \( r \geq 0 \) such that \( \lim_{n \to \infty} \int_0^{S(uTv)} \zeta(t) \, dt = r \).

If \( r > 0 \), then take limit for \( n \to \infty \), we get \( \psi(r) \leq F(\psi((r), \varphi(r)) \).
So \( \psi(r) = 0 \) or \( \varphi(r) = 0 \). Thus \( r = 0 \), which is a contradiction. Thus, we conclude that \( r = 0 \), that is,

\[
\lim_{n \to \infty} \int_0^1 \zeta(t) dt = 0,
\]

since for each \( \varepsilon > 0 \)
\( \int_0^\varepsilon \zeta(t) dt > 0 \), implies
\( \lim_{n \to \infty} S(u_n, u_n, u_{n+1}) = 0. \)

Now we show that the sequence \( \{u_n\} \) is a Cauchy sequence.

Assume that \( \{u_n\} \) is not Cauchy. Then there exists an \( \varepsilon > 0 \) and subsequences \( \{m_k\} \) and \( \{n_k\} \) such that \( m_k < n_k < m_{k+1} \) with

\[
S(u_{m_k}, u_{m_k}, u_{n_k}) \geq \varepsilon
\]

and

\[
S(u_{m_k}, u_{m_k}, u_{n_k-1}) < \varepsilon
\]

Hence using Lemma (1.3), we have

\[
S(u_{m_k-1}, u_{m_k-1}, u_{n_k-1}) \leq 2S(u_{m_k-1}, u_{m_k-1}, u_{n_k}) + S(u_{n_k-1}, u_{n_k-1}, u_{m_k}) < 2S(u_{m_k-1}, u_{m_k-1}, u_{m_k}) + \varepsilon
\]

and

\[
\lim_{k \to \infty} \int_0^1 \zeta(t) dt \leq \int_0^\varepsilon \zeta(t) dt
\]

Using the inequalities (2.1), (2.4) and (2.6) we obtain

\[
\psi(\int_0^\varepsilon \zeta(t) dt) \leq \psi( \int_0^1 \zeta(t) dt) \leq F(\psi(\int_0^1 \zeta(t) dt), \varphi(\int_0^1 \zeta(t) dt) ) \leq F(\psi(\int_0^\varepsilon \zeta(t) dt), \varphi(\int_0^\varepsilon \zeta(t) dt) )
\]

So \( \psi(\int_0^\varepsilon \zeta(t) dt) = 0 \) or \( \varphi(\int_0^\varepsilon \zeta(t) dt) = 0 \). Thus \( \int_0^\varepsilon \zeta(t) dt = 0 \), which is a contradiction with our assumption. So the sequence \( \{u_n\} \) is Cauchy. Using the completeness hypothesis, there exists \( w \in X \) such that

\[
\lim_{n \to \infty} T^n u_0 = w.
\]

From the inequality (2.1) we find

\[
\psi(\int_0^1 \zeta(t) dt) \leq F(\psi(\int_0^1 \zeta(t) dt), \varphi(\int_0^1 \zeta(t) dt) )
\]

If we take limit for \( n \to \infty \), we get
The function \( \psi \) or \( \varphi \). Consequently, the fixed point \( w \in S \). Thus \( \psi(\int_0^1 \zeta(t)dt) = 0 \) or \( \varphi(\int_0^1 \zeta(t)dt) = 0 \).

Thus \( \int_0^1 \zeta(t)dt = 0 \), which implies that \( S(Tw, Tw, w) = 0 \). Thus \( Tw = w \).

Now we show the uniqueness of the fixed point. Suppose that \( w_1 \) is another fixed point of \( T \). Using the inequality (2.1) we have

\[
\psi(\int_0^1 \zeta(t)dt) = \psi(\int_0^1 \zeta(t)dt) \leq F(\psi(\int_0^1 \zeta(t)dt), \varphi(\int_0^1 \zeta(t)dt))
\]

So \( \psi(\int_0^1 \zeta(t)dt) = 0 \) or \( \varphi(\int_0^1 \zeta(t)dt) = 0 \). Thus \( \int_0^1 \zeta(t)dt = 0 \).

Using \( \int_0^1 \zeta(t)dt > 0 \) we get \( w = w_1 \). Consequently, the fixed point \( w \) is unique.

Choosing \( F(s, t) = hs, \ 0 < h < 1, \psi(t) = t, \) in theorem (2.1) we have

**Corollary 2.2** [4] Let \((X, S)\) be a complete \( S \)-metric space \( h \in (0, 1) \), the function \( \zeta : P \to P \) be defined as for each \( \varepsilon > 0, \int_0^\varepsilon \zeta(t)dt > 0 \) and \( T : X \to X \) be a self-mapping of \( X \) such that

\[
\int_0^1 \zeta(t)dt \leq h \int_0^1 \zeta(t)dt
\]

for all \( u, v \in X \). Then \( T \) has a unique fixed point \( w \in X \) and we have \( \lim_{n \to \infty} T^n u = w \), for each \( u \in X \).

**Example 2.3** Let \( X = R, \ k = 10 \) be a fixed real number and function \( S : X \times X \times X \to [0, \infty) \) be defined as

\[
S(u, v, z) = \frac{z}{k + 1}(|v - z| + |v + z - 2u|)
\]

for all \( u, v, z \in R \). It can be seen that the function \( S \) is an \( S \)-metric. Now we show that \( S \)-metric can not be generated by metric \( \rho \). On the contrary, we assume that there exists a metric \( \rho \) such that

\[
S(u, v, z) = \rho(u, z) + \rho(v, z)
\]

for all \( u, v, z \in R \).
\[ \rho(u, z) = \frac{10}{11} |u - z| \]  

Similarly, we have

\[ S(v, v, z) = 2 \rho(v, z) = \frac{20}{11} (|v - z| + |v + z - 2u|) \]  
and

\[ \rho(v, z) = \frac{10}{11} |v - z| \]  

Using the equalities above equation (2.8), (2.9) and (2.10) we obtain

\[ \frac{10}{11} (|v - z| + |v + z - 2u|) = \frac{10}{11} |u - z| + \frac{10}{11} |v - u| \]

which is a contradiction, \( S \) is not generated by any metric and \((R, S)\) is a complete \( S \)-metric space. \( T : R \rightarrow R \) and \( Tu = \frac{u}{4} \) for all \( u \in R \) \( \zeta : P \rightarrow P \) where \( P = (0, \infty) \) as \( \zeta(t) = 2t \)

Let \( F(s, t) = s - t \) for all \( s, t \in [0, \infty) \). Also define \( \varphi, \psi : [0, \infty) \rightarrow [0, \infty) \) by \( \psi(t) = t \) and \( \varphi(t) = \frac{t}{2} \)

\[ F(\psi(\int_0^s \zeta(t) dt), \varphi(\int_0^s \zeta(t) dt)) = \psi(\int_0^s \zeta(t) dt) - \varphi(\int_0^s \zeta(t) dt) \]  

From equation (2.11), we have

\[ F(\psi(\int_0^\varepsilon \zeta(t) dt), \varphi(\int_0^\varepsilon \zeta(t) dt)) = \psi(\int_0^\varepsilon \zeta(t) dt) - \varphi(\int_0^\varepsilon \zeta(t) dt) \]

\[ = \psi(\int_0^{2\varepsilon} dt) - \varphi(\int_0^{2\varepsilon} dt) \]

\[ = \varepsilon^2 - \frac{\varepsilon^2}{2} > 0 \]

for all \( \varepsilon > 0 \), \( T \) satisfies the inequalities (2.1).

\[ \frac{100}{4(121)} |u - v|^2 \leq \frac{4 \times 100}{121} |u - v|^2 \quad \forall u, v \in R \]

\( T \) has a unique fixed point \( u = 0 \).

**Theorem 2.4** Let \((X, S)\) be a complete \( S \)-metric space \( \psi : [0, \infty) \rightarrow [0, \infty) \) is an altering distance function, \( \varphi \in \Phi_u \) and \( F \in C \), the function \( \zeta : P \rightarrow P \) be defined as for each \( \varepsilon > 0 \)

\[ \int_0^\varepsilon \zeta(t) dt > 0 \]  
and \( T : X \rightarrow X \) be a self-mapping of \( X \) such that

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\[
\psi( \int_0 ^{S(Tu, Tv)} \zeta(t)dt ) \leq F(\psi(h_1) \int_0 ^{S(Tu, Tv)} \zeta(t)dt + h_2 \int_0 ^{S(Tu, Tv)} \zeta(t)dt + h_3 \int_0 ^{S(Tu, Tv)} \zeta(t)dt \]
\[+ h_4 \int_0 ^{S(Tu, Tv)} \zeta(t)dt, \varphi(h_1) \int_0 ^{S(Tu, Tv)} \zeta(t)dt + h_2 \int_0 ^{S(Tu, Tv)} \zeta(t)dt \]
\[\max \{S(Tu, Tv), S(TTv, u)\} \]
\[\int_0 ^{S(TTv, u)} \zeta(t)dt, \varphi(h_1) \int_0 ^{S(TTv, u)} \zeta(t)dt + h_2 \int_0 ^{S(TTv, u)} \zeta(t)dt \]
\[\max \{S(TTv, u), S(TTv, v)\} \]
\[\int_0 ^{S(TTv, v)} \zeta(t)dt \]

\[\text{for all } u, v \in X \text{ with non negative real numbers } h_i (i \in \{1,2,3,4\}) \text{ satisfying } \max \{h_1 + 3h_4, h_1 + h_2 + h_3\} = 1. \text{ Then } T \text{ has a unique fixed point } w \in X \text{ and we have } \lim_{n \to \infty} T^n u = w, \text{ for each } u \in X. \]

**Proof.** Let \( u_0 \in X \) and the sequence \( \{u_n\} \) be defined as \( \lim_{n \to \infty} T^n u_0 = u_n \) Suppose that \( u_n \neq u_{n+1} \) for all \( n \). Using the inequality (2.12), the condition (S2) and Lemma 1.3 we get

\[
\psi( \int_0 ^{S(u_n, u_{n+1})} \zeta(t)dt ) = \psi( \int_0 ^{S(TTv, u)} \zeta(t)dt) \leq F(\psi(h_1) \int_0 ^{S(u_n, u_{n+1})} \zeta(t)dt + h_2 \int_0 ^{S(u_n, u_{n+1})} \zeta(t)dt \]
\[+ h_3 \int_0 ^{S(u_n, u_{n+1})} \zeta(t)dt + h_4 \int_0 ^{S(u_n, u_{n+1})} \zeta(t)dt, \varphi(h_1) \int_0 ^{S(u_n, u_{n+1})} \zeta(t)dt + h_2 \int_0 ^{S(u_n, u_{n+1})} \zeta(t)dt \]
\[\max \{S(u_n, u_{n+1}), S(TTv, u)\} \]
\[\int_0 ^{S(TTv, u)} \zeta(t)dt, \varphi(h_1) \int_0 ^{S(TTv, u)} \zeta(t)dt + h_2 \int_0 ^{S(TTv, u)} \zeta(t)dt \]
\[\max \{S(TTv, u), S(TTv, v)\} \]
\[\int_0 ^{S(TTv, v)} \zeta(t)dt \]

\[= F(\psi(h_1) \int_0 ^{S(u_{n+1}, u_{n+2})} \zeta(t)dt + h_2 \int_0 ^{S(u_{n+1}, u_{n+2})} \zeta(t)dt \]
\[+ h_3 \int_0 ^{S(u_{n+1}, u_{n+2})} \zeta(t)dt + h_4 \int_0 ^{S(u_{n+1}, u_{n+2})} \zeta(t)dt, \varphi(h_1) \int_0 ^{S(u_{n+1}, u_{n+2})} \zeta(t)dt + h_2 \int_0 ^{S(u_{n+1}, u_{n+2})} \zeta(t)dt \]
\[\max \{S(u_{n+1}, u_{n+2}), S(TTv, u)\} \]
\[\int_0 ^{S(TTv, u)} \zeta(t)dt, \varphi(h_1) \int_0 ^{S(TTv, u)} \zeta(t)dt + h_2 \int_0 ^{S(TTv, u)} \zeta(t)dt \]
\[\max \{S(TTv, u), S(TTv, v)\} \]
\[\int_0 ^{S(TTv, v)} \zeta(t)dt \]
\[ F(h_1) \leq F(h_1 + h_3) \]
\[ F(h_1 + h_3 + h_4) \]
\[ = F((h_1 + h_3 + h_4) \int_0^\infty \zeta(t) dt + (2h_3 + h_4) \int_0^\infty \zeta(t) dt, \varphi(h_1 + h_3 + h_4) \int_0^\infty \zeta(t) dt \]
\[ \varphi(h_1 + h_3 + h_4) \int_0^\infty \zeta(t) dt + (2h_3 + h_4) \int_0^\infty \zeta(t) dt \]

which implies
\[ \int_0^\infty \zeta(t) dt \leq \frac{h_1 + h_3 + h_4}{1 - 2h_3 - h_4} \int_0^\infty \zeta(t) dt \]

Since \[ \int_0^\infty \zeta(t) dt > 0, \]

so there exists \( r \geq 0 \) such that \[ \lim_{n \to \infty} \int_0^\infty \zeta(t) dt = r. \]

If \( r > 0 \), then take limit for \( n \to \infty \), we get \[ \psi(r) \leq F(\psi(r), \varphi(r)) \]
So \( \psi(r) = 0 \) or \( \varphi(r) = 0 \). Thus \( r = 0 \), which is a contradiction. Thus, we conclude that \( r = 0 \), that is, \[ \lim_{n \to \infty} \int_0^\infty \zeta(t) dt = 0, \]

since for each \( \varepsilon > 0 \), \[ \int_0^\infty \zeta(t) dt > 0, \]

implies
\[ \lim_{n \to \infty} S(u_n, u_n, u_n) = 0. \]

By the similar arguments used in the proof of Theorem (2.1), we see that the sequence \( \{u_n\} \) is Cauchy. Then there exists \( w \in X \) such that \[ \lim_{n \to \infty} T^n u_0 = w, \]
since \( (X, S) \) is a complete \( S \)-metric space. From the inequality (2.12) we find...
\[ \psi(\int_0^1 \zeta(t) \, dt) = \psi(\int_0^1 \zeta(t) \, dt) \leq F(\psi(h_1) \int_0^1 \zeta(t) \, dt + h_2 \int_0^1 \zeta(t) \, dt + h_3 \int_0^1 \zeta(t) \, dt) \]

Taking limit for \( n \to \infty \) and using Lemma 1.3 we get
\[ \psi(\int_0^1 \zeta(t) \, dt) \leq F(\psi(h_1) \int_0^1 \zeta(t) \, dt, \psi(h_2) \int_0^1 \zeta(t) \, dt + h_3 \int_0^1 \zeta(t) \, dt) \]
\[ \leq \psi(h_1 + h_2) \int_0^1 \zeta(t) \, dt \leq \psi(\int_0^1 \zeta(t) \, dt) \]

So \( \psi(h_1 + h_2) \int_0^1 \zeta(t) \, dt = 0 \) or \( \psi(h_1 + h_2) \int_0^1 \zeta(t) \, dt = 0 \).

Thus \( \int_0^1 \zeta(t) \, dt = 0 \), which implies that \( S(Tw, Tw) = 0 \). Thus \( Tw = w \). Now we show the uniqueness of the fixed point. Let \( w_1 \) be another fixed point of \( T \). Using the inequality (2.12) and Lemma 1.3, we get
\[ \psi(\int_0^1 \zeta(t) \, dt) = \psi(\int_0^1 \zeta(t) \, dt) \leq F(\psi(h_1) \int_0^1 \zeta(t) \, dt + h_2 \int_0^1 \zeta(t) \, dt + h_3 \int_0^1 \zeta(t) \, dt) \]

which implies
\[
\psi(\int_0^h \zeta(t) \, dt) \leq F(\psi((h_1 + h_2 + h_3) \int_0^h \zeta(t) \, dt), \varphi((h_1 + h_2 + h_3) \int_0^h \zeta(t) \, dt))
\]
\[
\leq \psi((h_1 + h_2 + h_3) \int_0^h \zeta(t) \, dt)
\]
\[
\leq \psi(\int_0^h \zeta(t) \, dt)
\]

So \( \psi((h_1 + h_2 + h_3) \int_0^h \zeta(t) \, dt) = 0 \) or \( \varphi((h_1 + h_2 + h_3) \int_0^h \zeta(t) \, dt) = 0 \). Then we obtain
\[
\int_0^h \zeta(t) \, dt = 0
\]

that is, \( w = w_1 \) since \( h_1 + h_2 + h_3 < 1 \). Consequently, \( T \) has a unique fixed point \( w \in X \).

Choosing \( F(s, t) = hs, \) \( 0 < h < 1, \) \( \psi(t) = t, \) (replace \( h_i \) with \( hh_i \)) in Theorem (2.4) we have

**Corollary 2.5** [4]Let \((X, S)\) be a complete \( S \)-metric space \( h \epsilon (0,1) \), the function \( \zeta : P \rightarrow P \) be defined as for each \( \epsilon > 0, \int_0^\epsilon \zeta(t) \, dt > 0 \) and \( T : X \rightarrow X \) be a self-mapping of \( X \) such that
\[
\int_0^{S(Tu, Tu, Tv)} \zeta(t) \, dt \leq h_1 \int_0^{S(u, u, v)} \zeta(t) \, dt + h_2 \int_0^{S(Tv, Tv, u)} \zeta(t) \, dt + h_3 \int_0^{S(Tu, Tu, v)} \zeta(t) \, dt
\]
\[
+ h_4 \int_0^{S(Tu, u, Tu), S(Tv, v, u))} \zeta(t) \, dt
\]
for all \( u, v \in X \) with non negative real numbers \( h_i (i \in \{1,2,3,4\}) \) satisfying \( \max\{h_1 + 3h_2 + 2h_4, h_1 + h_2 + h_3\} < 1 \). Then \( T \) has a unique fixed point \( w \in X \) and we have
\[
\lim_{n \to \infty} T^n u = w, \text{ for each } u \in X.
\]

**Example 2.6** Let \( X = R \) be the complete \( S \)-metric space with \( S \)-metric space defined in example (2.3). Let us define the self mapping \( T : R \rightarrow R \) as
\[
Tu = \begin{cases} 
2u + 39 & u \in (0,3) \\
90 & \text{otherwise}
\end{cases}
\]
for all \( u \in R \) and define a function \( \zeta : P \rightarrow P \) where \( P = (0, \infty) \) as \( \zeta(t) = 2t \)
\[
\int_0^\epsilon \zeta(t) \, dt = \int_0^\epsilon 2tdt = \epsilon^2 > 0 \quad \epsilon > 0.
\]

\( T \) satisfy the inequality (2.12) in theorem (2.4) for \( h_1 = h_2 = h_3 = 0, h_4 = \frac{1}{2} \) and the inequality (??) in theorem (2.7) for \( h_1 = h_1 = h_5 = 0, h_2 = \frac{1}{3} \). Hence \( T \) has a unique fixed point \( 90 \). But
\( T \) does not satisfy the inequality (2.1) in theorem (2.1). Indeed, if we take \( u = 0 \) and \( v = 1 \), then we obtain
\[
\psi(\int_0^{10} 2dt) = 100 \leq F(h(\int_0^3 2dt), \varphi(\int_0^3 2dt)) \leq \psi(h(\int_0^3 2dt)) \leq 9h
\]
which is a contradiction since \( h \in (0,1) \)

**Theorem 2.7** Let \((X,S)\) be a complete \(S\)-metric space \( \psi : [0,\infty) \to [0,\infty) \) is an altering distance function, \( \varphi \in \Phi_u \) and \( F \in \mathcal{C} \), the function \( \zeta : P \to P \) be defined as for each \( \varepsilon \in [0, \int_0^\varepsilon \zeta(t)dt] \) and \( T : X \to X \) be a self-mapping of \( X \) such that

\[
\begin{align*}
S(Tu, Tu, Tv) & \leq F(Uu, Vv) + \psi(\int_0^3 2dt) + \varphi(\int_0^3 2dt) + h_3 \\
S(Tu, Tu, u) & \leq \psi(\int_0^3 2dt + h_1) + \varphi(\int_0^3 2dt + h_2)
\end{align*}
\]

\[
\begin{align*}
\max \{S(u, u, v), S(Tu, Tu, u), S(Tu, Tu, v), S(Tv, Tv, u), S(Tv, Tv, v), S(Tv, Tu, v), S(Tv,Tv,v), S(Tv, Tu, v), S(Tv, Tv, v), S(Tv, Tv, v)\}
\end{align*}
\]

\[
\varphi(h_1) + \psi(\int_0^3 2dt + h_2) + \varphi(\int_0^3 2dt + h_3) + \psi(\int_0^3 2dt + h_4)
\]

\[
\max \{S(u, u, v), S(Tu, Tu, u), S(Tu, Tu, v), S(Tv, Tv, u), S(Tv, Tv, v), S(Tv, Tu, v), S(Tv, Tv, v)\}
\]

for all \( u, v \in X \) with non negative real numbers \( h_i (i \in \{1, 2, 3, 4, 5, 6\}) \) satisfying \( h_1 + h_2 + 3h_4 + h_5 + h_6 + h_7 + h_8 + h_9 = 1 \). Then \( T \) has a unique fixed point \( w \in X \) and we have \( \lim_{n \to \infty} T^n u = w \), for each \( u \in X \).

**Proof.** Let \( u_0 \in X \) and the sequence \( \{u_n\} \) be defined as \( \lim_{n \to \infty} T^n u_0 = u_n \). Suppose that \( u_n \neq u_{n+1} \) for all \( n \). Using the inequality (2.12), the condition (S2) and Lemma 1.3 we get
\[ \psi( \int_0^{u_{n+1}} \zeta(t) \, dt ) = \psi( \int_0^{u_{n-1}} \zeta(t) \, dt ) \]

\[ \leq F( \psi(h_1 + h_2) + \psi(h_4) + \psi(h_5) + \psi(h_6) ) \]

\[ + h_4 \int_0^{u_{n-1}} \zeta(t) \, dt + h_5 \int_0^{u_{n-1}} \zeta(t) \, dt + h_6 \int_0^{u_{n-1}} \zeta(t) \, dt \]

\[ \max \{ S(u_{n-1}^{1/n-1} \cdot u_{n+1}/u_{n-1} \cdot u_{n+1}/u_{n-1}) \cdot S(u_{n-1}^{1/n} \cdot u_{n+1}/u_{n-1} \cdot u_{n+1}/u_{n-1}) \cdot S(u_{n-1}^{1/n+1} \cdot u_{n+1}/u_{n-1} \cdot u_{n+1}/u_{n-1}) \} \]

which implies

\[ \int_0^{u_{n+1}} \zeta(t) \, dt \leq \frac{h_1 + h_2 + h_4 + h_6}{1 - 2h_4 - h_5 - 2h_6} \int_0^{u_{n-1}} \zeta(t) \, dt = h \int_0^{u_{n-1}} \zeta(t) \, dt \]

(2.15)

Since \( \int_0^{u_{n+1}} \zeta(t) \, dt > 0 \), so there exists \( r \geq 0 \) such that

\[ \lim_{n \to \infty} \int_0^{u_{n+1}} \zeta(t) \, dt = r. \]

If \( r > 0 \), then taking limit for \( n \to \infty \), we get \( \psi(r) \leq F(\psi(r), \phi(r)) \)

So \( \psi(r) = 0 \) or \( \phi(r) = 0 \). Thus \( r = 0 \), which is a contradiction. Thus, we conclude that \( r = 0 \), that is,

\[ \lim_{n \to \infty} \int_0^{u_{n+1}} \zeta(t) \, dt = 0, \]
since for each $\varepsilon > 0$, $\int_0^\varepsilon \zeta(t)dt > 0$, implies $\lim_{n\to\infty} S(u_n, u_{n+1}) = 0$.

By the similar arguments used in the proof of Theorem (2.1), we see that the sequence \{u_n\} is Cauchy. Then there exists $w \in X$ such that $\lim_{n\to\infty} T^n u_0 = w$, since $(X, S)$ is a complete $S$-metric space. From the inequality (??) we find

$$\psi(\int_0^\infty \zeta(t)dt) \leq F(\psi(h_1 \int_0^\infty \zeta(t)dt + h_2 \int_0^\infty \zeta(t)dt) + h_3 \int_0^\infty \zeta(t)dt + h_4 \int_0^\infty \zeta(t)dt + h_5 \int_0^\infty \zeta(t)dt)
\leq \psi((h_4 + h_5 + h_6) \int_0^\infty \zeta(t)dt)
\leq \psi(\int_0^\infty \zeta(t)dt)
\leq \psi(\int_0^\infty \zeta(t)dt)$$

If we take limit for $n \to \infty$, using Lemma 1.3 we get

$$\psi(\int_0^\infty \zeta(t)dt) \leq F(\psi(h_1 \int_0^\infty \zeta(t)dt + h_2 \int_0^\infty \zeta(t)dt) + h_3 \int_0^\infty \zeta(t)dt + h_4 \int_0^\infty \zeta(t)dt + h_5 \int_0^\infty \zeta(t)dt)
\leq \psi((h_4 + h_5 + h_6) \int_0^\infty \zeta(t)dt)
\leq \psi(\int_0^\infty \zeta(t)dt)
\leq \psi(\int_0^\infty \zeta(t)dt)$$

So $\psi((h_4 + h_5 + h_6) \int_0^\infty \zeta(t)dt) = 0$ or $\varphi((h_4 + h_5 + h_6) \int_0^\infty \zeta(t)dt) = 0$. Thus $S(Tw, Tw, w) = 0$, which implies that $S(Tw, Tw, w) = 0$. Thus $Tw = w$. Now we show the uniqueness of the fixed point. Let $w_1$ be another fixed point of $T$. Using the inequality (??) and Lemma 1.3, we get
\[
\psi\left( \int_{0}^{\infty} \zeta(t) dt \right) = \psi\left( \int_{0}^{\infty} \zeta(t) dt \right)
\]

\[
\leq F(\psi(h_{1} + \int_{0}^{\infty} \zeta(t) dt), \phi(h_{1} + \int_{0}^{\infty} \zeta(t) dt))
\]

\[
\leq \psi(\int_{0}^{\infty} \zeta(t) dt) \leq \psi(\int_{0}^{\infty} \zeta(t) dt)
\]

So \( \psi((h_{1} + h_{3} + h_{4} + h_{6}) \int_{0}^{\infty} \zeta(t) dt) = 0 \) or \( \phi((h_{1} + h_{3} + h_{4} + h_{6}) \int_{0}^{\infty} \zeta(t) dt) = 0 \). Then we obtain

\[
\int_{0}^{\infty} \zeta(t) dt = 0
\]

that is, \( w = w_{1} \) since \( h_{1} + h_{3} + h_{4} + h_{6} < 1 \). Consequently, \( T \) has a unique fixed point \( w \in X \).

**REFERENCES**


237-243

