1. INTRODUCTION

Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [12]. However, it was not until 1977, following an article [3] by Ernie Cockayne and Stephen Hedetniemi, that domination in graphs became an area of study by many researchers. One type of domination parameter is the secure domination in graphs. This was studied and introduced by E.J. Cockayne et.al [3,4]. Secure dominating sets can be applied as protection strategies by minimizing the number of guards to secure a system so as to be cost effective as possible. Other type of domination parameter is the restrained domination number in a graph. This was introduced by Telle and Proskurowski [10] indirectly as a vertex partitioning problem. One practical application of restrained domination is that of prisoners and guards. Here, each vertex not in the restrained dominating set corresponds to a position of a prisoner, and every vertex in the restrained dominating set corresponds to a position of a guard. To effect security, each prisoner’s position is observed by a guard’s position. To protect the rights of prisoners, each prisoners’ position is seen by at least one other prisoner’s position. To be cost effective, it is desirable to place a few guards as possible. In cite[7,8], Enriquez and Canoy, introduced the concepts of secure convex and restrained convex domination in graphs. In [11], Pushpam and Suseendran paper’s “Secure Restrained Domination in Graphs” studied few properties of secure restrained domination number of certain classes of graphs and evaluate $\gamma_{rs}(G)$ values for trees, unicyclic graphs, split graphs and generalized Petersen graphs. In papers [1,5,6,13,14], more results on secure and restrained domination in graphs where shown. In this paper, we characterize the secure restrained dominating sets in the join and corona of two graphs and give some important results. For other graph concepts, readers may refer to [9].

A graph $G$ is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the vertex-set of $G$ and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply $uv$) of distinct elements from $V(G)$ called the edge-set of $G$. The elements of $V(G)$ are called vertices and the cardinality $|V(G)|$ of $V(G)$ is the order of $G$. The elements of $E(G)$ are called edges and the cardinality $|E(G)|$ of $E(G)$ is the size of $G$. If $|V(G)| = 1$, then $G$ is called a trivial graph. If $E(G) = \emptyset$, then $G$ is called an empty graph. The open neighborhood of a vertex $v \in V(G)$ is the set $N_e(v) = \{u \in V(G) : uv \in E(G)\}$. The elements of $N_e(v)$ are called neighbors of $v$. The closed neighborhood of $v \in V(G)$ is the set $N_c(v) = \{u \in V(G) : uv \in E(G)\}$ of $v$. If $X \subseteq V(G)$, the open neighborhood of $X$ in $G$ is the set $N_e(X) = \bigcup_{v \in X} N_e(v)$. The closed neighborhood of $X$ in $G$ is the set $N_c(X) = \bigcup_{v \in X} N_c(v) \cup X$. When no confusion arises, $N_e(x)$ [resp. $N_c(x)$] will be denoted by $N(x)$ [resp. $\overline{N}(x)$].

A subset $S$ of $V(G)$ is a dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The domination number $\gamma(G)$ of $G$ is the smallest cardinality of a dominating set of $G$. A dominating set $S$ of $V(G)$ is a secure dominating set of $G$ if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G$. The minimum cardinality of a secure dominating set of $G$, denoted by $\gamma_r(G)$, is called the secure domination number of $G$. A secure dominating set of cardinality $\gamma_r(G)$ is called a $\gamma_r$-set of $G$. A set $S \subseteq V(G)$ is a restrained dominating set.
if every vertex not in \( S \) is adjacent to a vertex in \( S \) and to a vertex in \( V(G) \setminus S \). Alternately, a subset \( S \) of \( V(G) \) is a restrained dominating set if \( \mathcal{N}[S] = V(G) \) and \( (V(G) \setminus S) \) is a subgraph without isolated vertices. A restrained dominating set \( S \) of \( V(G) \) is a restrained secure dominating set of \( G \) if for each \( u \in V(G) \setminus S \), there exists \( v \in S \) such that \( uv \in E(G) \) and the set \( (S \setminus \{ v \}) \cup \{ u \} \) is a dominating set of \( G \). The minimum cardinality of a restrained secure dominating set of \( G \), denoted by \( \gamma_{rs}(G) \), is called the restrained secure domination number of \( G \). A restrained secure dominating set of cardinality \( \gamma_{rs}(G) \) is called a \( \gamma_{rs} \)-set of \( G \). Unless otherwise stated, all graphs in this paper are assumed to be simple and connected.

2. RESULTS

From the definitions, the following remark is immediate.

**Remark 2.1** Let \( G \) be any connected graph of order \( n \geq 3 \). Then

\[(i) \; \gamma(G) \leq \gamma_r(G) \leq \gamma_{rs}(G); \text{ and} \]
\[(ii) \; \gamma_{rs}(G) \in \{1,2,\ldots,n-3,n-2,n\}.\]

The next result is the characterization of restrained secure dominating set with restrained secure domination number one.

**Theorem 2.2** Let \( G \) be a connected graph of order \( n \geq 3 \). Then \( \gamma_{rs}(G) = 1 \) if and only if \( G = K_n \).

**Proof:** Clearly, \( \gamma_{rs}(K_n) = 1 \). Suppose now that \( \gamma_{rs}(G) = 1 \). Let \( S = \{x\} \) be a restrained secure dominating set in \( G \). Suppose that \( G \) is not complete. Then there exist \( y,z \in V(G) \) such that \( yz \in E(G) \). It follows that \( (S \setminus \{x\}) \cup \{y\} \neq E(G) \). This implies that \( S \) is not a restrained secure dominating set, contrary to our assumption. Thus, \( G = K_n \). \( \square \)

The following theorem shows that every integers \( k \) and \( n \) with \( 1 \leq k \leq n \) is realizable as restrained secure domination number and order of \( G \) respectively.

**Theorem 2.3** Given positive integers \( k \) and \( n \) such that \( n \geq 3 \) and \( k \in \{1,2,\ldots,n-2,n\} \), there exists a connected graph \( G \) with \( |V(G)| = n \) and \( \gamma_{rs}(G) = k \).

**Proof:** Consider the following cases:

Case 1. Suppose \( k = 1 \).

Let \( G = K_n \). Then \( |V(G)| = n \) and \( \gamma_{rs}(G) = 1 \).

Case 2. Suppose \( k = n \).

Let \( G = K_{1,n-1} \). Then \( |V(G)| = n \) and \( \gamma_{rs}(G) = k \).

Case 3. Suppose \( 2 \leq k \leq n - 2 \).

Let \( r = n - k \). Consider the graph \( G = (\{v\}) + (K_r \cup \tilde{K}_{k-1}) \) as shown in Figure 1.

![Figure 1: A graph G with |V(G)| = n and \( \gamma_{rs}(G) = k \)](image)

The set \( S = \{v,a_1,a_2,\ldots,a_{k-1}\} \) is a minimum restrained secure dominating set of \( G \). Hence, \( |V(G)| = r + (k - 1) + 1 = n \) and \( \gamma_{rs}(G) = (k - 1) + 1 = k \). This proves the assertion. \( \square \)

**Corollary 2.4** The difference \( \gamma_{rs}(G) - \gamma(G) \) can be made arbitrarily large.
Proof: Let $n$ be a positive integer. By Theorem 2.3, there exists a connected graph $G$ such that $\gamma_{rs}(G) = n + 1$ and $\gamma(G) = 1$. Thus, $\gamma_{rs}(G) - \gamma(G) = n$. Therefore, $\gamma_{rs}(G) - \gamma(G)$ can be made arbitrarily large. ■

**Theorem 2.5** Let $G$ be connected graph of order $n \geq 4$. If $G = K_2 + H$ where $H$ is a non-complete graph and has no isolated vertices, then $\gamma_{rs}(G) = 2$.

Proof: Suppose $G = K_2 + H$, where $H$ is non-complete and has no isolated vertices. Let $V(K_2) = \{a,b\} = S$. Then $S$ is a dominating set of $G$. Let $v \in V(H)$. Then $S_v = (S \setminus \{a\}) \cup \{v\} = \{b,v\}$. Since $bu \in E(G)$ for all $u \in V(H)$ and $vx \in E(G)$ for all $x \in S$, it follows that $N_G[S_v] = V(G)$ and hence $S_v$ is a dominating set of $G$. Thus, $S$ is a secure dominating set of $G$. Since $(V(G) \setminus S) = H$ has no isolated vertices, it follows that $S$ is a restrained dominating set of $G$. Accordingly $S$ is a restrained secure dominating set of $G$ and hence $\gamma_{rs}(G) \leq 2$. Since $G$ is non-complete, $\gamma_{rs}(G) \geq 2$ by Theorem 2.2. Therefore, $\gamma_{rs}(G) = 2$. ■

The *join* of two graphs $G$ and $H$ is the graph $G + H$ with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv: u \in V(G), v \in V(H)\}$.

A nonempty subset $S$ of $V(G)$, where $G$ is any graph, is a *clique* set in $G$ if the graph $(S)$ induced by $S$ is complete. The following result characterized the restrained secure dominating sets in the join of two graphs.

**Theorem 2.6** Let $G$ and $H$ be connected non-complete graphs. Then a proper subset $S$ of $V(G + H)$ is a restrained secure dominating set in $G + H$ if and only if one of the following statements holds:

(i) $S$ is a secure dominating set of $G$.

(ii) $S$ is a secure dominating set of $H$.

(iii) $S = S_G \cup S_H$ where $S_G = \{v\} \subset V(G)$ and $S_H = \{w\} \subset V(H)$ and

a) $S_G$ is a dominating set of $G$ and $S_H$ is a dominating set of $H$.

b) $S_G$ is dominating set of $G$ and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique set in $H$; or

c) $S_H$ is dominating set of $H$ and $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique set in $G$; or

d) $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique set in $G$ and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique set in $H$.

(iv) $S = S_G \cup S_H$ where $S_G \subseteq V(G)$ and $S_H = \{w\} \subset V(H)$ and $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique set in $G$.

(v) $S = S_G \cup S_H$ where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ and $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique set in $G$.

(vi) $S = S_G \cup S_H$ where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ and $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique set in $G$.

Proof: Suppose that $S$ is a restrained secure dominating set of $G + H$. Consider the following cases:

Case 1. Suppose that $S \subseteq V(G)$ or $S \subseteq V(H)$. If $S \subseteq V(G)$, then $S$ is a secure dominating set of $G$. This shows that statement (i) holds. Similarly, if $S \subseteq V(H)$, then statement (ii) holds.

Case 2. Suppose $S = S_G \cap V(G) \neq \emptyset$ and $S_H = S \cap V(H) \neq \emptyset$. Then $S = S_G \cup S_H$. Consider the following subcases.

**Subcase 1.** Suppose that $S_G = \{v\} \subset V(G)$ is a dominating set of $G$ and $S_H = \{w\} \subset V(H)$ is a dominating set of $H$. Then we are done with statement (iii). Suppose that $S_G$ is a dominating set of $G$ and $S_H$ is not a dominating set of $H$. Let $x \in (V(G) \setminus S_G) \setminus N_G(S_G)$. Since $S$ is a dominating set of $G + H$, $\{w,x\}$ is a dominating set in $G + H$ (and hence in $H$). Since $wx \not\in E(H)$, $xy \in E(H)$ for every $y \not\in N_H(w)$. This implies that $y \in (V(H) \setminus S_H) \setminus N_H(S_H)$. Since $x$ was arbitrarily chosen, it follows that the subgraph $(V(H) \setminus S_H) \setminus N_H(S_H)$ is complete. Hence, $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique set in $H$. This proves statement (iii). Similarly, if $S_H$ is dominating set of $H$ and $S_G$ is not a dominating set of $G$, then statement (iii) holds by following similar arguments in statements (iiib) and (iiic).

**Subcase 2.** Suppose that $S_G \subseteq V(G)$ and $S_H = \{w\} \subset V(H)$. If $S_G$ is a dominating set of $G$, then statement (i) holds. Suppose that $S_G$ is not a dominating set of $G$. Let $x \in (V(G) \setminus S_G) \setminus N_G(S_G)$. Since $S$ is a dominating set of $G + H$, $\{x\}$ is a dominating set in $G + H$ (and hence in $G$). Since $wx \not\in E(G)$ for every $x \in S_G$, $xy \in E(G)$ for every $y \not\in N_G(S_G)$ (otherwise, $S_G$ is not dominating set in $G + H$). This implies that $y \in (V(G) \setminus S_G) \setminus N_G(S_G)$. Since $x$ was arbitrarily chosen, it follows that the subgraph $(V(G) \setminus S_G) \setminus N_G(S_G)$ is complete. Hence, $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique set in $G$. This proves statement (iv). Similarly, statement (v) holds, if $S_G = \{v\} \subset V(G)$ and $S_H \subseteq V(H)$ and $(S_H \setminus w) \geq 2$.

**Subcase 3.** Suppose that $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$. Let $|S_G| \geq 2$. If $S_G$ is a dominating set of $G$, then statement (i) holds. Suppose that $S_G$ is not a dominating set of $G$. If $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in $G$, then statement (iv) holds. Suppose that $(V(G) \setminus S_G) \setminus N_G(S_G)$ is not a clique in $G$. If $|S_H| = 1$, say $S_H = \{w\}$, then there exists $x \in (V(G) \setminus S_G)$
$N_G(S_G)$ such that $S_G = (S \setminus \{w\}) \cup \{x\}$ is not a dominating set of $G$(and hence in $G + H$). This contradict to our assumption that $S$ is a secure dominating set of $G + H$. Thus, $|S_H| \geq 2$. Similarly, if $|S_H| \geq 2$ and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is not a clique in $H$, then $|S_G| \geq 2$. This proves statement $(vi)$.

For the converse, suppose first that statement $(i)$ holds. Let $u \in V(G + H)$. If $u \in V(G)$, then there exists $v \in S \cap N_G(u)$ such that $S_u = (S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G$ (and hence $S_u$ is a dominating set of $G + H$). Suppose that $u \in V(H)$. Since $|S| \geq 2, N_{G + H}(S_u) = N_{G + H}(S \setminus \{v\}) \cup N_{G + H}(u) = V(G + H)$. Thus $S_u$ a dominating set of $G$ and hence of $G + H$. Accordingly, $S$ is a secure dominating set of $G + H$. Now, let $x \in V(G + H) \setminus S$. Then $xy \in E(G + H)$ for all $y \in V(H)$. Since $x, y \in V(G + H) \setminus S$, it follows that the subgraph induced by $V(G + H) \setminus S$ has no isolated vertices and hence $S$ is a restrained dominating set of $G$. Accordingly, $S$ is a restrained secure dominating set of $G + H$. Similarly, if statement $(ii)$ holds, $S$ is a restrained secure dominating set of $G + H$.

Suppose that statement $(iii)$ holds. Then $S = \{v, w\}$ is a dominating set of $G + H$. Let $x \in V(G + H) \setminus S$. Then $vx \in E(G + H)$ and $S_x = (S \setminus \{v\}) \cup \{x\} = \{w, x\}$ is a dominating set of $G + H$, that is, $S$ is a secure dominating set of $G + H$. Since $G$ and $H$ are connected non-complete graphs, $V(G) \setminus \emptyset$ and $V(H) \setminus \emptyset$. Let $x \in V(G) \setminus \{v\}$ and $y \in V(H) \setminus \{w\}$. Then $xy \in E(G + H)$. Since $x, y \in V(G + H) \setminus S$, it follows that $S$ is a restrained dominating set of $G$. Accordingly, $S$ is a restrained secure dominating set of $G + H$.

Suppose that statement $(iv)$ holds. Since $S_G = \{v\}$ is a dominating set of $G$ (and hence of $G + H$), $S = S_G \cup S_H$ is a dominating set of $G + H$. Let $u \in V(G + H) \setminus S$. Then $uv \in E(G + H)$ and $S_u = (S \setminus \{v\}) \cup \{u\} = \{w, u\}$. If $u \in V(G)$, then $S_u$ is a dominating set of $G + H$. Suppose that $u \in V(H)$. Then $u \in N_H(w)$, and $u \in (V(H) \setminus S_G) \cup N_H(S_H)$. Since $(V(H) \setminus S_G) \cup N_H(S_H)$ is a clique in $H$, it follows that $N_H(S_u) = N_H[w] \cup N_H[u] = V(H)$. Thus, $S_u$ is a dominating set of $H$ and hence of $G + H$. Accordingly, $S$ is a secure dominating set of $G + H$. Since $G$ and $H$ are connected non-complete graphs, by similar arguments used above, $S$ is a restrained secure dominating set of $G + H$. Similarly, $S$ is a restrained secure dominating set of $G + H$ if $(ii)$ holds.

Suppose that statement $(iii)$ holds. Then $S = \{v, w\}$ is a dominating set of $G + H$. Let $u \in V(G + H) \setminus S$. Let $u \in V(G)$. If $u \in N_G(S_G)$, the $uv \in E(G)$ and $S_u = (S \setminus \{v\}) \cup \{u\} = \{w, u\}$ is a dominating set of $G + H$. If $u \notin N_G(S_G)$, then $u \in (V(G) \setminus S_G) \cup N_G(S_G)$. Since $(V(G) \setminus S_G) \cup N_G(S_G)$ is a clique in $G$, it follows that $uv \in E(G + H)$ and $S_u = (S \setminus \{w\}) \cup \{u\} = \{v, u\}$ is a dominating set of $G$ and hence of $G + H$. Similarly, if $u \in V(H)$ then $S_u = (S \setminus \{v\}) \cup \{u\} = \{w, u\}$ is a dominating set of $H$ and hence of $H + H$. Thus, $S$ is a secure dominating set of $G + H$. Now, $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ implies that $V(G) \setminus S_G \neq \emptyset$ and $V(H) \setminus S_H \neq \emptyset$. Let $x \in V(G) \setminus S_G$ and $y \in V(H) \setminus S_H$. Then $xy \in E(G + H)$. Since $x, y \in V(G + H) \setminus S$, it follows that $S$ is a restrained dominating set of $G + H$. Accordingly, $S$ is a restrained secure dominating set of $G + H$.

Similarly, $S$ is a restrained secure dominating set of $G + H$ if any of the following statements $(iv), (v)$, or $(vi)$ holds. ■

The following result is a quick consequence of Theorem 2.6.

**Corollary 2.7** Let $G$ and $H$ be connected non-complete graphs and let $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$. If $\gamma(G) = 1 = \gamma(H)$ or $\gamma_G(G) = 2$ or $\gamma_H(H) = 2$ then $\gamma_{G + H}(G + H) = 2$.

The corona of two graphs $G$ and $H$ is the graph $G \circ H$ obtained by taking one copy of $G$ and $m$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex of the $i$th copy of $H$. The join of vertex $v$ of $G$ and a copy $H^v$ of $H$ in the corona of $G$ and $H$ is denoted by $v + H^v$.

**Remark 2.8** Let $G$ and $H$ be nontrivial connected graphs. A nonempty subset $S$ of $V(G \circ H)$ is a dominating set of $G \circ H$ if and only if $V(G) \subseteq S$ or $\bigcup_{v \in V(G)} S_v \subseteq S$ where for each $v \in V(G)$, $S_v$ is a dominating set of $H^v$.

The following result characterize the restrained secure dominating sets in the corona of two connected graphs.

**Theorem 2.9** Let $G$ and $H$ be nontrivial connected graphs. A nonempty subset $S$ of $V(G \circ H)$ is a restrained secure dominating set of $G \circ H$ if and only if for each $v \in V(G)$, one of the following is satisfied.

(i) $S = V(G)$ and $H$ is complete.

(ii) $S = V(G) \cup (\bigcup_{v \in V(G)} S_v)$, where for each $v \in V(G)$, $S_v$ is a dominating set of $H^v$ and $(V(H^v) \setminus S_v)$ has no isolated vertices.

(iii) $S = \bigcup_{v \in V(G)} S_v$, where for each $v \in V(G)$, $S_v$ is a dominating set of $H^v$.

**Proof:** Suppose that a nonempty subset $S$ of $V(G \circ H)$ is an restrained secure dominating set of $G \circ H$. Since $S$ is a dominating set, in view of Remark 2.8, $V(G) \subseteq S$ or $\bigcup_{v \in V(G)} S_v \subseteq S$ where for each $v \in V(G)$, $S_v$ is a dominating set of $H^v$. Consider the following cases.
Case 1. Suppose that $V(G) \subseteq S$. Let $S = V(G)$. If $H$ is non-complete, then there exist distinct vertices $x, y \in V(H)$ such that $xy \notin E(H)$. This implies that for each $v \in V(G)$, $(S \setminus \{v\}) \cup \{x\}$ is not a dominating set of $G \ast H$ contrary to our assumption that $S$ is a secure dominating set of $G \ast H$. Thus, $H$ is complete. This proves statement (i). Let $S \not\in V(G)$. If $S \subseteq V(G)$ then $S$ is not a dominating set of $G \ast H$. This implies that $V(G) \subseteq S$. Let $S = V(G) \cup \{u\}$, where for each $v \in V(G), S_v$ is a dominating set of $H^v$. If for each $v \in V(G), (V(H^v) \setminus S_v)$ has isolated vertices, then $\{V(G \ast H) \setminus S\} = \{V(G \ast H) \setminus \bigcup_{v \in V(G)} V(H^v) \setminus S_v\}$ has isolated vertices contrary to our assumption that $S$ is a restrained dominating set of $G \ast H$. Hence, $(V(H^v) \setminus S_v)$ has no isolated vertices. This proves statement (ii).

Case 2. Suppose that $V(G) \not\subseteq S$. Then $S = \bigcup_{v \in V(G)} S_v$. Since $S$ is a dominating set of $G \ast H$, $S_v$ must be a dominating set in $H^v$ for each $v \in V(H^v)$. Let $x \in S$ and $y \in V(G \ast H) \setminus S$. Then $x \subseteq S_v$ and $y \in V(H^v) \setminus S_v$ for some $u \in V(G)$. Since $S$ is a secure dominating set of $G \ast H$, $S^* = (S \setminus \{x\}) \cup \{y\} = \bigcup_{v \in V(G)} \bigcup_{S_v} \{y\} \cup \{y\} = \bigcup_{v \in V(G)} \bigcup_{S_v} \{y\} \cup \{y\}$ is a dominating set of $G \ast H$. This implies that $(S_v \setminus \{x\}) \cup \{y\}$ is a dominating set of $H^u$ for some $u \in V(G)$. Since $S_v$ is a dominating set of $H^u$, it follows that $S_v$ is a secure dominating set of $H^u$ for some $u \in V(G)$. Consequently, for each $v \in V(G), S_v$ is a secure dominating set of $H^u$. This proves statement (iii).

For the converse, suppose that for each $v \in V(G)$, statement (i) or (ii) or (iii) holds. First, if statement (i) holds, then $S = V(G)$ is a dominating set of $G \ast H$ by Remark 2.8. Now, let $u \in V(G \ast H) \setminus S$. Since $H$ is complete, for each $v \in V(G), v \in S \cap N_{G \ast H}(u)$ and $S_v = S \setminus \{v\} \cup \{u\}$ is a dominating set of $G \ast H$. This implies that $S$ is a secure dominating set of $G \ast H$. Further, $H$ is a nontrivial connected graphs. Thus, $(V(G \ast H) \setminus S) = (V(G \ast H) \setminus V(G \ast H))$ has no isolated vertices. Hence, $S$ is a restrained dominating set of $G \ast H$. Accordingly, $S$ is a restrained secure dominating set of $G \ast H$.

Suppose that statement (ii) holds. Clearly, $S$ is a dominating set of $G \ast H$. Let $u \in V(G \ast H) \setminus S$. Then $xu \in E(G \ast H)$ for some $x \in S$.

Case 1. Consider that $x \in V(G)$. Then $S^* = (S \setminus \{x\}) \cup \{u\} = \bigcup_{v \in V(G)} (V(G) \cup (U_{v \in V(G)} S_v) \setminus \{x\}) \cup \{u\} = \bigcup_{v \in V(G)} (V(G) \setminus \{x\}) \cup \{u\} \cup \bigcup_{v \in V(G)} S_v$.

Since $S_v$ is a dominating set of $H^v$ for all $x \in V(G)$, it follows that $U_{v \in V(G)} S_v$ is a dominating set of $G \ast H$ and hence $S^*$ is a dominating set of $G \ast H$. Thus $S$ is a secure dominating set of $G \ast H$.

Case 2. Consider that $x \not\in V(G)$. Then $x \in \bigcup_{v \in V(G)} S_v$. This implies that $x \subseteq S_v^r$ for some $v^r \in V(G)$. Thus $S^* = (S \setminus \{x\}) \cup \{u\} = \bigcup_{v \in V(G)} (V(G) \cup (U_{v \in V(G)} S_v) \setminus \{x\}) \cup \{u\} = \bigcup_{v \in V(G)} (V(G) \cup (U_{v \in V(G)} S_v) \cup (S_v - \{x\})) \cup \{u\}$.

Since $V(G)$ is a dominating set of $G \ast H$, it follows that $S^*$ is a dominating set of $G \ast H$. Thus, $S$ is a secure dominating set of $G \ast H$.

Now, $(V(H^v) \setminus S_v)$ has no isolated vertices, it follows that $(\bigcup_{v \in V(G)} V(H^v) \setminus S_v)$ has no isolated vertices. Thus, $(V(G \ast H) \setminus S) = \bigcup_{v \in V(G)} (V(G \ast H) \setminus (V(G) \cup U_{v \in V(G)} S_v)) = \bigcup_{v \in V(G)} V(H^v) \setminus S_v$ has no isolated vertices. This implies that $S$ is a restrained dominating set of $G \ast H$. Accordingly, $S$ is a restrained secure dominating set of $G \ast H$.

Suppose that statement (iii) holds. Then $S$ is a dominating set of $G \ast H$ by Remark 2.8. Let $u$ be an element of $V(G \ast H) \setminus S$. Consider the following cases:

Case 1. If $u \in V(G)$, then $ux \in E(u + H^u) \subseteq E(G \ast H)$ for each $x \in S_u$. Since $u$ dominate $u + H^u$, it follows that $(S_u \setminus \{x\}) \cup \{u\}$ is a dominating set of $u + H^u$. Thus, $(S \setminus \{x\}) \cup \{u\} = (U_{v \in V(G)} S_v) \cup (S_u \setminus \{x\}) \cup \{u\}$ is a dominating set of $G \ast H$. Hence, $S$ is a secure dominating set of $G \ast H$. Now, if $S_v = V(H^v)$ for each $v \in V(G)$, then $(V(G \ast H) \setminus S) = G$. Since $G$ is a nontrivial connected graph, it follows that $(V(G \ast H) \setminus S)$ has no isolated vertices. Thus, $S$ is a restrained dominating set of $G \ast H$. If $S_v \subseteq V(H^v)$ for each $v \in V(G)$, then let $u \in V(H^u) \setminus S_u$. This implies that $uu \in E(H^u)$ for each $u \in V(G)$. 

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Since $u$ and $u'$ are arbitrary elements of $V(G \circ H) \setminus S$, it follows that $(V(G \circ H) \setminus S)$ has no isolated vertices. Hence $S$ is a restrained dominating set of $G \circ H$. Accordingly, $S$ is a restrained secure dominating set of $G \circ H$.

Case 2. If $u \notin V(G)$, then $u \in V(H^v) \setminus S_v$ for each $v \in V(G)$. Since for each $v \in V(G), S_v$ is a secure dominating set of $H^v$, it follows that $ux \in E(H^v)$ for some $x \in S_v$ such that $(S_v \setminus \{x\}) \cup \{u\}$ is a dominating set of $H^v$. By similar arguments used above, $(S \setminus \{x\}) \cup \{u\}$ is a dominating set of $G \circ H$ and hence $S$ is a secure dominating set of $G \circ H$. It is clear that $(V(G \circ H) \setminus S)$ has no isolated vertices and hence, $S$ is a restrained secure dominating set of $G \circ H$.

Corollary 2.10 Let $G$ be a connected graph and $H$ is complete of order $n \geq 2$. Then $\gamma_{rs}(G \circ H) = |V(G)|$.

Corollary 2.11 Let $G$ be a connected graph and $S_v$ is a $\gamma_{rs}$-set in $H^v$ for each $v \in V(G)$. Then $\gamma_{rs}(G \circ H) = \sum_{v \in V(G)} |S_v|$.

REFERENCES