ON ARITHMETIC OPERATIONS OF PENTAGONAL FUZZY NUMBERS WITH THE $\alpha$ –CUT METHOD

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Abstract: In this paper new arithmetic operations on $\alpha$-cut of Pentagonal fuzzy numbers are studied. Some important properties are also proved with the aid of $\alpha$-cut of Pentagonal fuzzy numbers. Relevant examples are also included to illustrate the result obtained.

Keywords: Fuzzy Number, Pentagonal Fuzzy Number, $\alpha$-cut of pentagonal Fuzzy Number.

1. INTRODUCTION

Decision making problems in the real world are very often uncertain (or) vague in most cases. In 1965, Zadeh [8] introduced the concept of fuzzy set theory to meet those problems. Fuzzy set theory permits the gradual assessment of the membership of elements in a set which is described in the interval [0, 1]. It can be used in a wide range of domain where information is incomplete and imprecise. Interval arithmetic was suggested by means of Zadeh’s extension principle [8,9], the usual Arithmetic operations on real numbers can be extended to the ones defined on Fuzzy numbers. Dubois and Prade [1] has defined any of the fuzzy numbers as a fuzzy subset of the real line [2, 3, 4, 5, 6, 10].

Stephen Dinagar et al. [7] discussed Arithmetic Operations of Hexagonal Fuzzy Numbers using the $\alpha$-cut Method. In this work some new elementary operators on $\alpha$ –cuts of Pentagonal fuzzy numbers have been introduced. In this paper, section 2 contains some basic definitions needed for this work. In section 3 we deal with $\alpha$ –cut of Pentagonal fuzzy number. In section 4 we present new arithmetic operations on $\alpha$ –cut of Pentagonal fuzzy numbers. In section 5 we present some examples related to our results. In section 6 we prove properties by using the proposed arithmetic operations of $\alpha$ –cut of Pentagonal fuzzy numbers. Finally, conclusion is included in section 7.

2. PRELIMINARIES

Definition 2.1 (Fuzzy set)

A Fuzzy set $\tilde{A}$ is defined by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)): x \in A, \mu_{\tilde{A}}(x) \in [0,1]\}$. In the pair $(x, \mu_{\tilde{A}}(x))$, the first element $x$ belong to the classical set $A$, the second element $\mu_{\tilde{A}}(x)$, belong to the interval $[0,1]$, called Membership function.

Definition 2.2 (Convex fuzzy set)

A fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\} \subseteq X$ is called convex fuzzy set if all $A_{\alpha}$ are convex set (i.e.) for every element $x_1 \in A_{\alpha}$ and $x_2 \in A_{\alpha}$ for every $\alpha \in [0.1]$, $\lambda x_1 + (1 - \lambda)x_2 \in A_{\alpha}$ for all $\lambda \in [0,1]$. Otherwise the fuzzy set is called non-convex fuzzy set.
Definition 2.3 (Fuzzy number)

A fuzzy set $\tilde{A}$, defined on the set of real number $R$ is said to be fuzzy number if it has the following characteristics

(i) $\tilde{A}$ is normal
(ii) $\tilde{A}$ is convex set
(iii) The support of $\tilde{A}$ is closed and bounded.

Definition 2.4 (Pentagonal fuzzy number)

A fuzzy number $\tilde{A}_p$ is pentagonal fuzzy number denoted by $\tilde{A}_p = (a_1, a_2, a_3, a_4, a_5)$, where $a_1, a_2, a_3, a_4, a_5$ are real numbers and its membership function $\mu_{\tilde{A}_p}(x)$ is given by

$$
\mu_{\tilde{A}_p}(x) = \begin{cases} 
0, & x < a_1 \\
\frac{1}{2} \left[ \frac{x-a_1}{a_2-a_1} \right], & a_1 \leq x \leq a_2 \\
\frac{1}{2} + \frac{1}{2} \left[ \frac{x-a_2}{a_3-a_2} \right], & a_2 \leq x \leq a_3 \\
1, & x = a_3 \\
\frac{1}{2} + \frac{1}{2} \left[ \frac{a_4-x}{a_5-a_4} \right], & a_3 \leq x \leq a_4 \\
\frac{1}{2} \left[ \frac{a_5-x}{a_5-a_4} \right], & a_4 \leq x \leq a_5 \\
0, & x > a_5
\end{cases}
$$

Definition 2.5

A pentagonal fuzzy number can be defined as $\tilde{A}_p = (M_1(x), J_1(x), J_2(x), M_2(x))$ for $x \in [0,1]$ where,

(i) $M_1(x)$ is strictly increasing continuous function on $[0,0.5]$
(ii) $J_1(x)$ is strictly increasing continuous function on $[0.5,1]$
(iii) $J_2(x)$ is strictly decreasing continuous function on $[1,0.5]$
(iv) $M_2(x)$ is strictly decreasing continuous function on $[0.5,0]$

Remark 2.6.

The pentagonal fuzzy number $\tilde{A}_p$ becomes triangular fuzzy number if $a_2 - a_1 = a_4 - a_3$. 
2.7. Arithmetic Operations on Pentagonal Fuzzy Numbers

Let us consider \( \tilde{A}_p = (a_1, a_2, a_3, a_4, a_5) \) and \( \tilde{B}_p = (b_1, b_2, b_3, b_4, b_5) \) be two pentagonal fuzzy numbers then,

(i) Addition

\[ \tilde{A}_p + \tilde{B}_p = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5) \]

(ii) Subtraction

\[ \tilde{A}_p - \tilde{B}_p = (a_1 - b_5, a_2 - b_4, a_3 - b_3, a_4 - b_2, a_5 - b_1) \]

(iii) Multiplication

\[ \tilde{A}_p \times \tilde{B}_p = \left( \frac{a_1 b_1}{s}, \frac{a_2 b_2}{s}, \frac{a_3 b_3}{s}, \frac{a_4 b_4}{s}, \frac{a_5 b_5}{s} \right) \]

Where \( s = b_1 + b_2 + b_3 + b_4 + b_5 \) (or)

\[ \tilde{A}_p \times \tilde{B}_p = (a_1 \tilde{B}_p(b), a_2 \tilde{B}_p(b), a_3 \tilde{B}_p(b), a_4 \tilde{B}_p(b), a_5 \tilde{B}_p(b)) \]

Where \( \tilde{B}_p(b) = \left( \frac{b_1 + b_2 + b_3 + b_4 + b_5}{s} \right) \)

(iv) Division

\[ \tilde{A}_p \div \tilde{B}_p = \left( \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \frac{a_4}{b_4}, \frac{a_5}{b_5} \right) \]

Where \( s = b_1 + b_2 + b_3 + b_4 + b_5 \) (or)

\[ \tilde{A}_p \div \tilde{B}_p = \left( \frac{a_1}{\tilde{B}_p(b)}, \frac{a_2}{\tilde{B}_p(b)}, \frac{a_3}{\tilde{B}_p(b)}, \frac{a_4}{\tilde{B}_p(b)}, \frac{a_5}{\tilde{B}_p(b)} \right) \]

Where \( \tilde{B}_p(b) = \left( \frac{b_1 + b_2 + b_3 + b_4 + b_5}{s} \right) \)

(v) Join Operator

\[ \tilde{A}_p \vee \tilde{B}_p = (a_1 \vee b_1, a_2 \vee b_2, a_3 \vee b_3, a_4 \vee b_4, a_5 \vee b_5) \]
(vi) Meet Operator

\[ \tilde{A}_\alpha \wedge \tilde{B}_\alpha = (a_\alpha \wedge b_\alpha, a_2 \wedge b_2, a_3 \wedge b_3, a_4 \wedge b_4, a_5 \wedge b_5). \]

**Definition 2.8 (Ranking Function)**

We define a ranking function \( \tilde{R} : F(R) \rightarrow R \) which maps each fuzzy number to real line \( F(R) \) represent the set of all pentagonal fuzzy numbers. If \( R \) be any linear ranking functions.

\[ \tilde{R}(\tilde{A}_\alpha) = \left( \frac{s_1 + s_2 + s_3}{5} \right). \]

Also, we defined orders on \( F(R) \) by,

(i) \( \tilde{R}(\tilde{A}_\alpha) \geq \tilde{R}(\tilde{B}_\alpha) \) if and only if \( \tilde{A}_\alpha \geq \tilde{B}_\alpha \).

(ii) \( \tilde{R}(\tilde{A}_\alpha) \leq \tilde{R}(\tilde{B}_\alpha) \) if and only if \( \tilde{A}_\alpha \leq \tilde{B}_\alpha \).

(iii) \( \tilde{R}(\tilde{A}_\alpha) = \tilde{R}(\tilde{B}_\alpha) \) if and only if \( \tilde{A}_\alpha = \tilde{B}_\alpha \).

3. \( \alpha \) - CUTF OF PENTAGONAL FUZZY NUMBER

The crisp set \( A_\alpha \) called alpha cut is defined as

\[ A_\alpha = \{ x \in X | \mu_{A_\alpha}(x) \geq \alpha \} \]

\[ A_\alpha = \begin{cases} [M_1(\alpha), M_2(\alpha)] & \text{for } \alpha \in [0.05] \\ [J_1(\alpha), J_2(\alpha)] & \text{for } \alpha \in [0.51] \end{cases} \]

3.1 \( \alpha \) - Cut Operations

The interval \( A_\alpha \), for all \( \alpha \in [0,1] \), is obtained as follows:

Consider, \( M_1(\alpha) = \frac{1}{2} \left( \frac{x - s_1}{a_2 - a_1} \right) \)

Then, \( M_1(\alpha) = \alpha \)

(i.e.) \( M_1(\alpha) = 2\alpha (a_2 - a_1) + a_1 \).

Similarly, \( M_2(\alpha) = -2\alpha (a_5 - a_4) + a_5 \).

This implies

\[ [M_1(\alpha), M_2(\alpha)] = [2\alpha (a_2 - a_1) + a_1, -2\alpha (a_5 - a_4) + a_5] \]

Consider, \( J_1(\alpha) = \frac{1}{2} + \frac{1}{2} \left( \frac{x - s_2}{a_3 - a_2} \right) \)

Then, \( J_1(\alpha) = \alpha \)

(i.e.) \( J_1(\alpha) = 2\alpha (a_2 - a_1) + 2a_2 - a_2 \).

Similarly, \( J_2(\alpha) = -2\alpha (a_4 - a_3) + 2a_4 - a_2 \).
This implies
\[ [f_1(x), f_2(x)] = [2a_1(a_2 - a_1) + 2a_2 - a_2, -2a_1(a_2 - a_1) + 2a_2 - a_2] \]

Hence
\[ A_\alpha = \frac{[2a_1(a_2 - a_1) + a_2 - 2a_1(a_2 - a_1) + a_2]}{[2a_1(a_2 - a_1) + 2a_2 - a_2, -2a_1(a_2 - a_1) + 2a_2 - a_2]} \text{ for } \alpha \in [0,0.5] \\
B_\alpha = \frac{[2a_1(a_2 - a_1) + a_2 - 2a_1(a_2 - a_1) + a_2]}{[2a_1(a_2 - a_1) + 2a_2 - a_2, -2a_1(a_2 - a_1) + 2a_2 - a_2]} \text{ for } \alpha \in [0.5,1] 

4. NEW ARITHMETIC OPERATIONS ON PENTAGONAL FUZZY NUMBERS USING \( \alpha \)-CUT

The arithmetic operations between \( \alpha \)-cut of pentagonal fuzzy number are given below:

Let \( A_\alpha = (a_1, a_2, a_3, a_4, a_5) \) and \( B_\alpha = (b_1, b_2, b_3, b_4, b_5) \) be two pentagonal fuzzy numbers for all \( \alpha \in [0,1] \). We shall use interval arithmetic.

\[ A_\alpha = \frac{[2a_1(a_2 - a_1) + a_2 - 2a_1(a_2 - a_1) + a_2]}{[2a_1(a_2 - a_1) + 2a_2 - a_2, -2a_1(a_2 - a_1) + 2a_2 - a_2]} \text{ for } \alpha \in [0,0.5] \\
B_\alpha = \frac{[2a_1(b_2 - b_1) + b_1 - 2a_1(b_2 - b_1) + b_1]}{[2a_1(b_2 - b_1) + 2b_2 - b_2, -2a_1(b_2 - b_1) + 2b_2 - b_2]} \text{ for } \alpha \in [0.5,1] 

4.1 Addition of \( \alpha \)-Cuts of Two Pentagonal Fuzzy Numbers

\[ A_\alpha (+) B_\alpha = \begin{cases} 
2a_1[(a_2 + b_2) - (a_2 + b_2)] + (a_1 + b_1), & \text{for } \alpha \in [0,0.5] \\
-2a_1[(a_2 + b_2) - (a_2 + b_2)] + (a_1 + b_1), & \text{for } \alpha \in [0.5,1] 
\end{cases} 

4.2 Difference of \( \alpha \)-Cuts of Two Pentagonal Fuzzy Numbers

\[ A_\alpha (-) B_\alpha = \begin{cases} 
2a_1[(a_2 - b_2) - (a_2 - b_2)] + (a_1 - b_1), & \text{for } \alpha \in [0,0.5] \\
-2a_1[(a_2 - b_2) - (a_2 - b_2)] + (a_1 - b_1), & \text{for } \alpha \in [0.5,1] 
\end{cases} 

4.3 Multiplication of \( \alpha \)-Cuts of Two Pentagonal Fuzzy Numbers

\[ A_\alpha (\cdot) B_\alpha = \begin{cases} 
[(2a_1(a_2 - a_1) + a_1)(b_2), (2a_1(a_2 - a_1) + a_1)(b_2)], & \text{for } \alpha \in [0,0.5] \\
[(2a_1(a_2 - a_1) + a_1)(b_2), (2a_1(a_2 - a_1) + a_1)(b_2)], & \text{for } \alpha \in [0.5,1] 
\end{cases} 

Where,
\[ R(b) = \frac{a_1(b_3 - b_2) + b_3 + b_2}{b_3 + b_2} \text{ for } \alpha \in [0,0.5] \\
\[ R(b) = \frac{a_1(b_3 - b_2) + b_3 + b_2}{b_3 + b_2} \text{ for } \alpha \in [0.5,1] 

4.4 Division of \( \alpha \)-Cuts of Two Pentagonal Fuzzy Numbers

\[ A_\alpha (/) B_\alpha = \begin{cases} 
\left[\frac{[2a_1(a_2 - a_1) + a_1]}{R(b)}, \frac{[2a_1(a_2 - a_1) + a_1]}{R(b)}\right], & \text{for } \alpha \in [0,0.5] \\
\left[\frac{[2a_1(a_2 - a_1) + a_1]}{R(b)}, \frac{[2a_1(a_2 - a_1) + a_1]}{R(b)}\right], & \text{for } \alpha \in [0.5,1] 
\end{cases} 

Where,
4.5 Join Operator of \(\alpha\)-Cuts of Two Pentagonal Fuzzy Numbers

\[
\tilde{R}(b) = \frac{\max[(b_2 - b_1) - (b_a - b_b)] + b_1 + b_5}{\alpha} \quad \text{for } \alpha \in [0.0.5].
\]

\[
\tilde{R}(b) = \frac{\max[(b_3 - b_2) - (b_a - b_b)] - 2b_3 + 2(b_1 + b_4)}{\alpha} \quad \text{for } \alpha \in [0.5.1].
\]

4.5 Meet Operator of \(\alpha\)-Cuts of Two Pentagonal Fuzzy Numbers

\[
\tilde{A}_e(V)B_e = \begin{cases} 
\frac{\min[2\alpha(a_2 - a_1) + a_3, 2\alpha(b_2 - b_1) + b_3]}{\alpha} & \text{for } \alpha \in [0.0.5], \\
\frac{\max[2\alpha(a_3 - a_2) - a_3 + 2a_2, 2\alpha(b_3 - b_2) - b_3 + 2b_2]}{\alpha} & \text{for } \alpha \in [0.5.1].
\end{cases}
\]

5. NUMERICAL EXAMPLES

Example 5.1

Let \(A_p = (1.2.3.4.5)\) and \(B_p = (2.4.6.8.10)\) be two pentagonal fuzzy numbers (PFNs).

By the Arithmetic operations on PFNs. We have,

\[
A_p(+) B_p = (1.2.3.4.5)(+)(2.4.6.8.10).
\]

\[
A_p(+) B_p = (3.6.9.12.15).
\]

By the New arithmetic operations on \(\alpha\) –cut of Pentagonal fuzzy numbers. We have the same illustration numbers as,

\[
A_e(+) B_e = \begin{cases} 
\frac{2\alpha[(a_2 + b_2) - (a_1 + b_1)] + (a_1 + b_1)}{\alpha} & \text{for } \alpha \in [0.0.5], \\
\frac{2\alpha[(a_3 + b_3) - (a_2 + b_2)] + (a_2 + b_2)}{\alpha} & \text{for } \alpha \in [0.5.1].
\end{cases}
\]

\[
A_e(+) B_e = \begin{cases} 
[6\alpha + 3. -6\alpha + 15] & \text{for } \alpha \in [0.0.5], \\
[6\alpha + 3. -6\alpha + 15] & \text{for } \alpha \in [0.5.1].
\end{cases}
\]

Since for both \(\alpha \in [0.0.5]\) and \(\alpha \in [0.5.1]\) arithmetic intervals are equal.

Therefore, \(A_e(+) B_e = [6\alpha + 3. -6\alpha + 15]\) for all \(\alpha \in [0.1]\)

when \(\alpha = 0\), \(A_e(+) B_0 = [3.15]\).

when \(\alpha = 0.5\), \(A_{e,5}(+) B_{e5} = [6.12]\).

when \(\alpha = 1\), \(A_{e,1}(+) B_1 = [9.9]\).

Hence \(A_e(+) B_0 = [3.6.9.12.15]\). Thus all the points coincide with the sum of the two pentagonal fuzzy numbers.

Example 5.2
Let $\mathcal{A}_x = (1,2,3,4,5)$ and $\mathcal{B}_x = (2,4,6,8,10)$ be two fuzzy numbers.

By the Arithmetic operations on PFNs. We have,

$\mathcal{A}_x(-)\mathcal{B}_x = (1,2,3,4,5)(-)(2,4,6,8,10)$

$\mathcal{A}_x(-)\mathcal{B}_x = (-9,-6,-3,0,3)$.

By the New arithmetic operations on $\alpha-$cut of Pentagonal fuzzy numbers. We have the same illustration numbers as,

$A_x(-)B_x = \begin{cases} 
2\alpha [(a_2 - b_2) - (a_1 - b_1)] + (a_1 - b_1) & \text{for } \alpha \in [0,0.5] \\
-2\alpha [(a_3 - b_3) - (a_2 - b_2)] + (a_2 - b_2) & \text{for } \alpha \in [0.5,1]
\end{cases}$

$A_x(-)B_x = \begin{cases} 
[6\alpha - 9,-6\alpha + 3] & \text{for } \alpha \in [0,0.5] \\
[6\alpha - 9,-6\alpha + 3] & \text{for } \alpha \in [0.5,1]
\end{cases}$

Since for both $\alpha \in [0,0.5]$ and $\alpha \in [0.5,1]$ arithmetic intervals are equal.

Therefore, $A_x(-)B_x = [6\alpha - 9,-6\alpha + 3]$ for all $\alpha \in [0,1]$

when $\alpha = 0$, $A_x(-)B_x = [-9,3]$.

when $\alpha = 0.5$, $A_{x,5}(-)B_{x,5} = [-6,0]$.

when $\alpha = 1$, $A_x(-)B_1 = [-3,-3]$.

Hence $A_x(-)B_x = [-9,-6,-3,0,3]$. Thus all the points coincide with the difference of the two pentagonal fuzzy numbers.

**Example 5.3**

Let $\mathcal{A}_x = (1,2,3,4,5)$ and $\mathcal{B}_x = (2,4,6,8,10)$ be two fuzzy numbers.

By the Arithmetic operations on PFNs. We have,

$\mathcal{A}_x(\times)\mathcal{B}_x = (1,2,3,4,5)(\times)(2,4,6,8,10)$

$\mathcal{A}_x(\times)\mathcal{B}_x = (6,12,18,24,30)$.

By the New arithmetic operations on $\alpha-$cut of Pentagonal fuzzy numbers. We have the same illustration numbers as,$

$A_x(\times)B_x = \begin{cases} 
[2\alpha [(a_2 - a_2) - (a_1 - a_1)] + (a_1 - a_1)\bar{R}(b),(\alpha_2(a_3 - a_2) + a_3)\bar{R}(b)] & \text{for } \alpha \in [0,0.5] \\
[2\alpha [(a_2 - a_2) - (a_1 - a_1)] + (a_1 - a_1)\bar{R}(b),(\alpha_2(a_3 - a_2) + a_3)\bar{R}(b)] & \text{for } \alpha \in [0.5,1]
\end{cases}$

Where,

$\bar{R}(b) = \frac{2\alpha[(b_2-b_1)-(b_1-b_2)]+b_1+b_2}{\alpha}$, \text{ for } $\alpha \in [0,0.5]$.

$\bar{R}(b) = \frac{2\alpha[(b_2-b_1)-(b_1-b_2)]+b_1+b_2}{\alpha}$, \text{ for } $\alpha \in [0.5,1]$.

$A_x(\times)B_x = \begin{cases} 
[12\alpha + 6, -12\alpha + 30] & \text{for } \alpha \in [0,0.5] \\
[12\alpha + 6, -12\alpha + 30] & \text{for } \alpha \in [0.5,1]
\end{cases}$
Since for both $\alpha \in [0,0.5]$ and $\alpha \in [0.5,1]$ arithmetic intervals are equal.

Therefore, $A_\alpha (x)B_\alpha = [14\alpha + 7, -14\alpha + 49]$ for all $\alpha \in [0,1]$

when $\alpha = 0$, $A_0(x)B_0 = [6,30]$.

when $\alpha = 0.5$, $A_{0.5}(x)B_{0.5} = [12,24]$.

when $\alpha = 1$, $A_1(x)B_1 = [18]$.

Hence $A_\alpha (x)B_\alpha = [6,12,18,24,30]$. Thus all the points coincide with the multiplication of the two pentagonal fuzzy numbers.

**Example 5.4**

Let $A_\alpha = (1,2,3,4,5)$ and $B_\alpha = (2,4,6,8,10)$ be two fuzzy numbers.

By the Arithmetic operations on PFNs. We have,

$\bar{A}_\alpha B_\alpha = (1,2,3,4,5) \cdot (2,4,6,8,10)$.

$\bar{A}_\alpha (x)B_\alpha = \left\{ \frac{[2\alpha(a_1 - a_2) + a_3]}{\bar{R}(b)} , \frac{[-2\alpha(a_4 - a_2) + a_2]}{\bar{R}(b)} \right\}$ for $\alpha \in [0,0.5]$

$\bar{A}_\alpha (x)B_\alpha = \left\{ \frac{[2\alpha(a_1 - a_2) - a_2 + 2a_2]}{\bar{R}(b)} , \frac{[-2\alpha(a_4 - a_2) + 2a_4 - a_2]}{\bar{R}(b)} \right\}$ for $\alpha \in [0.5,1]$

Where,

$\bar{R}(b) = \frac{b_1b_2 - b_3b_4 + b_3b_5}{\alpha}$ for $\alpha \in [0,0.5]$.

$\bar{R}(b) = \frac{b_1b_2 - b_3b_4 - b_5b_6 + 2(b_5 + b_6)}{\alpha}$ for $\alpha \in [0.5,1]$.

$A_\alpha (x)B_\alpha = \left\{ \frac{[2\alpha + 1]}{6} , \frac{-2\alpha + 5}{6} \right\}$ for $\alpha \in [0,0.5]$

$A_\alpha (x)B_\alpha = \left\{ \frac{2\alpha + 1}{6} , \frac{-2\alpha + 5}{6} \right\}$ for $\alpha \in [0.5,1]$

Since for both $\alpha \in [0,0.5]$ and $\alpha \in [0.5,1]$ arithmetic intervals are equal.

Therefore, $A_\alpha (x)B_\alpha = \left[ \frac{2\alpha + 1}{6} , \frac{-2\alpha + 5}{6} \right]$ for all $\alpha \in [0,1]$

when $\alpha = 0$, $A_0(x)B_0 = [\frac{1}{6}, \frac{5}{6}]$.

when $\alpha = 0.5$, $A_{0.5}(x)B_{0.5} = [\frac{2}{6}, \frac{4}{6}]$.

when $\alpha = 1$, $A_1(x)B_1 = [\frac{3}{6}, \frac{3}{6}]$.

Hence $A_\alpha (x)B_\alpha = [\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}]$ Thus all the points coincide with the division of the two pentagonal fuzzy numbers.
Example 5.5

Let \( A_p = (1.2, 3, 4, 5) \) and \( B_p = (2.4, 6, 8, 10) \) be two fuzzy numbers.

By the Arithmetic operations on PFNs. We have,
\[
A_p \vee B_p = (1.2, 3, 4, 5) \vee (2.4, 6, 8, 10),
\]
\[
A_p \wedge B_p = (2.4, 6, 8, 10).
\]

By the New arithmetic operations on \( \alpha \)-cut of Pentagonal fuzzy numbers. We have the same illustration numbers as,
\[
A_p(\vee)B_p = \begin{cases} 
\max\{2\alpha(a_2 - a_1) + a_1, 2\alpha(b_2 - b_1) + b_1\} & \text{for } \alpha \in [0,0.5] \\
\max\{-2\alpha(a_3 - a_4) + a_5, -2\alpha(b_3 - b_4) + b_5\} & \text{for } \alpha \in [0.5,1]
\end{cases}
\]
\[
A_p(\wedge)B_p = \begin{cases} 
\max\{2\alpha + 1.4\alpha + 2\}, \max\{-2\alpha + 5, -4\alpha + 10\} & \text{for } \alpha \in [0,0.5] \\
\max\{2\alpha + 1.4\alpha + 2\}, \max\{-2\alpha + 5, -4\alpha + 10\} & \text{for } \alpha \in [0.5,1]
\end{cases}
\]

Since for both \( \alpha \in [0,0.5] \) and \( \alpha \in [0.5,1] \) arithmetic intervals are equal.

Therefore, \( A_p(\vee)B_p = [\max\{2\alpha + 1.4\alpha + 2\}, \max\{-2\alpha + 5, -4\alpha + 10\}] \) for all \( \alpha \in [0,1] \)

when \( \alpha = 0 \), \( A_p(\vee)B_0 = [2,10] \)

when \( \alpha = 0.5 \), \( A_p(\vee)B_{0.5} = [4,8] \).

when \( \alpha = 1 \), \( A_p(\vee)B_1 = [6,6] \).

Hence \( A_p(\vee)B_p = (2.4, 6, 8, 10) \). Thus all the points coincide with the join operation of the two pentagonal fuzzy numbers.

Example 5.6

Let \( A_p = (1.2, 3, 4, 5) \) and \( B_p = (2.4, 6, 8, 10) \) be two fuzzy numbers.

By the Arithmetic operations on PFNs. We have,
\[
A_p(\wedge)B_p = (1.2, 3, 4, 5) \wedge (2.4, 6, 8, 10),
\]
\[
A_p(\wedge)B_p = (1.2, 3, 4, 5).
\]

By the New arithmetic operations on \( \alpha \)-cut of Pentagonal fuzzy numbers. We have the same illustration numbers as,
\[
A_p(\wedge)B_p = \begin{cases} 
\min\{2\alpha(a_2 - a_1) + a_1, 2\alpha(b_2 - b_1) + b_1\} & \text{for } \alpha \in [0,0.5] \\
\min\{-2\alpha(a_3 - a_4) + a_5, -2\alpha(b_3 - b_4) + b_5\} & \text{for } \alpha \in [0.5,1]
\end{cases}
\]
\[
A_p(\wedge)B_p = \begin{cases} 
\min\{2\alpha + 1.4\alpha + 2\}, \min\{-2\alpha + 5, -4\alpha + 10\} & \text{for } \alpha \in [0,0.5] \\
\min\{2\alpha + 1.4\alpha + 2\}, \min\{-2\alpha + 5, -4\alpha + 10\} & \text{for } \alpha \in [0.5,1]
\end{cases}
\]
Since for both \(\alpha \in [0,0.5]\) and \(\alpha \in [0.5,1]\) arithmetic intervals are equal.

Therefore, \(A_\alpha(\land)B_\alpha = [\min[2\alpha + 1.4\alpha + 2], \min[-2\alpha + 5, -4\alpha + 10]]\) for all \(\alpha \in [0,1]\)

when \(\alpha = 0\), \(A_0(\land)B_0 = [1,5]\)

when \(\alpha = 0.5\), \(A_{0.5}(\land)B_{0.5} = [2,4]\).

when \(\alpha = 1\), \(A_1(\land)B_1 = [3,3]\).

Hence \(A_\alpha(\land)B_\alpha = [1,2,3,4,5]\). Thus all the points coincide with the meet operation of the two pentagonal fuzzy numbers.

### 6. PROPERTIES OF PENTAGONAL FUZZY NUMBERS USING \(\alpha\)-CUTS

**Property 6.1**

The sum of the \(\alpha\)-cuts of the two PFNs are equal to the \(\alpha\)-cut of their sum. (i.e., \((\hat{A}_F(+)\hat{B}_F)_\alpha = A_\alpha(+)B_\alpha\).**

**Proof**

Let \(\hat{A}_F = (a_1, a_2, a_3, a_4, a_5)\) and \(\hat{B}_F = (b_1, b_2, b_3, b_4, b_5)\) be two pentagonal fuzzy numbers. Then, the \(\alpha\) – Cuts of two pentagonal fuzzy numbers \(\hat{A}_F\) and \(\hat{B}_F\) are

\[
A_\alpha = \begin{cases}
\{[2\alpha(a_2 - a_1) + a_1, -2\alpha(a_5 - a_4) + a_5]\} & \text{for } \alpha \in [0,0.5] \\
\{2\alpha(a_3 - a_2) + 2a_2 - a_3, -2\alpha(a_4 - a_3) + 2a_4 - a_3\} & \text{for } \alpha \in [0.5,1]
\end{cases}
\]

\[
B_\alpha = \begin{cases}
\{2\alpha(b_2 - b_1) + b_1, -2\alpha(b_5 - b_4) + b_5\} & \text{for } \alpha \in [0,0.5] \\
\{2\alpha(b_3 - b_2) + 2b_3 - b_3, -2\alpha(b_4 - b_2) + 2b_4 - b_2\} & \text{for } \alpha \in [0.5,1]
\end{cases}
\]

We have to prove \((\hat{A}_F(+)\hat{B}_F)_\alpha = A_\alpha(+)B_\alpha\).

\[
A_\alpha(+B_\alpha) = \begin{cases}
\{2\alpha(a_2 - a_1) + a_2, 2\alpha(b_2 - b_2) + b_1\}, & \text{for } \alpha \in [0,0.5] \\
[-2\alpha(a_5 - a_4) + a_5 - 2\alpha(b_5 - b_4) + b_1], & \text{for } \alpha \in [0.5,1]
\end{cases}
\]

\[
\{2\alpha(a_3 - a_2) + 2a_3 - a_3, -2\alpha(b_3 - b_2) + 2b_3 - b_3\}, & \text{for } \alpha \in [0,0.5] \\
[-2\alpha(a_4 - a_2) + 2a_4 - a_2, -2\alpha(b_4 - b_3) + 2b_4 - b_3] & \text{for } \alpha \in [0.5,1]
\]

\(\Rightarrow (A_\alpha(+)B_\alpha)_\alpha\)

\(\Rightarrow (\hat{A}_F(+)\hat{B}_F)_\alpha = A_\alpha(+)B_\alpha\).

Hence \((\hat{A}_F(+)\hat{B}_F)_\alpha = A_\alpha(+)B_\alpha\).
Property 6.2

The difference of the $\alpha$-cuts of the two PFNs are equal to the $\alpha$-cut of their difference. (i.e.)
\[ (A_p(-)B_p)_\alpha = A_\alpha(-)B_\alpha. \]

**Proof**

Let $A_p = (a_1, a_2, a_3, a_4, a_5)$ and $B_p = (b_1, b_2, b_3, b_4, b_5)$ be two pentagonal fuzzy numbers. Then, the $\alpha$-Cuts of two pentagonal fuzzy numbers $A_p$ and $B_p$ are
\[
A_\alpha = \begin{cases} 
[2\alpha(a_3 - a_1) + a_5, -2\alpha(a_3 - a_4) + a_5] & \text{for } \alpha \in [0, 0.5) \\
[2\alpha(a_5 - a_3) + a_2 - a_4, -2\alpha(a_5 - a_2) + 2a_4 - a_2] & \text{for } \alpha \in [0.5, 1] 
\end{cases},
\]
\[
B_\alpha = \begin{cases} 
[2\alpha(b_3 - b_1) + b_5, -2\alpha(b_3 - b_4) + b_5] & \text{for } \alpha \in [0, 0.5) \\
[2\alpha(b_5 - b_3) + b_2 - b_4, -2\alpha(b_5 - b_2) + 2b_4 - b_2] & \text{for } \alpha \in [0.5, 1] 
\end{cases}.
\]

We have to prove $(A_p(-)B_p)_\alpha = A_\alpha(-)B_\alpha$.

\[
A_\alpha(-)B_\alpha = \begin{cases} 
[2\alpha(a_3 - a_1) + a_5, -2\alpha(a_3 - a_4) + a_5] & \text{for } \alpha \in [0, 0.5) \\
[2\alpha(a_5 - a_3) + a_2 - a_4, -2\alpha(a_5 - a_2) + 2a_4 - a_2] & \text{for } \alpha \in [0.5, 1] 
\end{cases} -
\]
\[
\begin{cases} 
[2\alpha(b_3 - b_1) + b_5, -2\alpha(b_3 - b_4) + b_5] & \text{for } \alpha \in [0, 0.5) \\
[2\alpha(b_5 - b_3) + b_2 - b_4, -2\alpha(b_5 - b_2) + 2b_4 - b_2] & \text{for } \alpha \in [0.5, 1] 
\end{cases}.
\]

Hence $(A_p(-)B_p)_\alpha = A_\alpha(-)B_\alpha$.

Property 6.3

The multiplication of the $\alpha$-Cuts of the two PFNs are equal to the $\alpha$-Cut of their multiplication. (i.e.)
\[ (A_p\times B_p)_\alpha = A_\alpha\times B_\alpha. \]

**Proof**

Let $A_p = (a_1, a_2, a_3, a_4, a_5)$ and $B_p = (b_1, b_2, b_3, b_4, b_5)$ be two pentagonal fuzzy numbers. Then, the $\alpha$-Cuts of two pentagonal fuzzy numbers $A_p$ and $B_p$ are
\[
A_\alpha = \begin{cases} 
[2\alpha(a_3 - a_1) + a_5, -2\alpha(a_3 - a_4) + a_5] & \text{for } \alpha \in [0, 0.5) \\
[2\alpha(a_5 - a_3) + a_2 - a_4, -2\alpha(a_5 - a_2) + 2a_4 - a_2] & \text{for } \alpha \in [0.5, 1] 
\end{cases},
\]
\[
B_\alpha = \begin{cases} 
[2\alpha(b_3 - b_1) + b_5, -2\alpha(b_3 - b_4) + b_5] & \text{for } \alpha \in [0, 0.5) \\
[2\alpha(b_5 - b_3) + b_2 - b_4, -2\alpha(b_5 - b_2) + 2b_4 - b_2] & \text{for } \alpha \in [0.5, 1] 
\end{cases}.
\]

We have to prove $(A_p\times B_p)_\alpha = A_\alpha\times B_\alpha$.
First, we take
\[
\left( \tilde{A}_p \times \tilde{B}_p \right) = \left( a_1 \tilde{R}(b), a_2 \tilde{R}(b), a_3 \tilde{R}(b), a_4 \tilde{R}(b), a_5 \tilde{R}(b) \right).
\]
Where \( \tilde{R}(b) = \frac{(b_1+b_2+b_3+b_4+b_5)}{5} \).

\[
\left( \tilde{A}_p \times \tilde{B}_p \right) = \left( a_1 \tilde{R}(b), a_2 \tilde{R}(b), a_3 \tilde{R}(b, a_4 \tilde{R}(b), a_5 \tilde{R}(b) \right) \alpha.
\]

\[
\begin{align*}
\left[ 2\alpha \left( a_1 \tilde{R}(b) - a_5 \tilde{R}(b) \right) + a_4 \tilde{R}(b) \right] & \text{ for } \alpha \in [0,0.5] \\
\left[ 2\alpha \left( a_5 \tilde{R}(b) - a_1 \tilde{R}(b) \right) + 2a_4 \tilde{R}(b) - a_5 \tilde{R}(b), - a_4 \tilde{R}(b) \right] & \text{ for } \alpha \in [0.5,1]
\end{align*}
\]

\[
\begin{align*}
\left[ 2\alpha \left( a_3 \tilde{R}(b) - a_1 \tilde{R}(b) \right) + a_4 \tilde{R}(b) \right] & \text{ for } \alpha \in [0,0.5] \\
\left[ 2\alpha \left( a_4 \tilde{R}(b) - a_3 \tilde{R}(b) \right) + a_5 \tilde{R}(b) \right] & \text{ for } \alpha \in [0.5,1]
\end{align*}
\]

Next, we take
\[
A_p(\times)B_p = \left\{ \begin{array}{l}
\left[ 2\alpha(a_2 - a_1) + a_1, -2\alpha(a_5 - a_3) + a_4 \right] \\
\left[ 2\alpha(a_5 - a_3) + 2a_4 - 2a_2 - 2a_4 - a_1 \right] \\
\left[ 2\alpha(b_2 - b_1) + b_1, -2\alpha(b_5 - b_3) + b_3 \right] \\
\left[ 2\alpha(b_5 - b_3) + 2b_2 - 2b_5 - 2b_2 + b_1 \right] \end{array} \right. 
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
\left[ 2\alpha(a_2 - a_1) + a_1, -2\alpha(a_5 - a_3) + a_4 \tilde{R}(b) \right] \\
\left[ 2\alpha(a_5 - a_3) + 2a_4 - 2a_2 - 2a_4 - a_1 \tilde{R}(b) \right] \\
\left[ 2\alpha(b_2 - b_1) + b_1, -2\alpha(b_5 - b_3) + b_3 \tilde{R}(b) \right] \\
\left[ 2\alpha(b_5 - b_3) + 2b_2 - 2b_5 - 2b_2 + b_1 \tilde{R}(b) \right] \end{array} \right. 
\]

where,
\[
\tilde{R}(b) = \frac{[2a\alpha(b_3 - b_2)] - [2a\alpha(b_2 - b_1)] + b_1 + b_3}{\alpha} \text{ for } \alpha \in [0,0.5].
\]

\[
\tilde{R}(b) = \frac{[2a\alpha(b_2 - b_3)] - [2a\alpha(b_3 - b_4)] + b_4 + b_2}{\alpha} \text{ for } \alpha \in [0.5,1].
\]

Finally, we have, \( \left( \tilde{A}_p \times \tilde{B}_p \right) = A_p(\times)B_p \).

**Property 6.4**

The division of the \( \alpha \)-Cuts of the two PFNs are equal to the alpha cut of their division. (i.e.) \( \left( \tilde{A}_p / \tilde{B}_p \right) = A_p / B_p \).

**Proof**

Let \( \tilde{A}_p = (a_0, a_2, a_3, a_4, a_5) \) and \( \tilde{B}_p = (b_0, b_2, b_3, b_4, b_5) \) be two pentagonal fuzzy numbers. Then, the \( \alpha \)-Cuts of two pentagonal fuzzy numbers \( \tilde{A}_p \) and \( \tilde{B}_p \) are
\[
\begin{align*}
A_p = \left\{ \begin{array}{l}
\left[ 2\alpha(a_2 - a_1) + a_1, -2\alpha(a_5 - a_3) + a_4 \right] \\
\left[ 2\alpha(a_5 - a_3) + 2a_4 - 2a_2 - 2a_4 - a_1 \right] \end{array} \right. 
\text{ for } \alpha \in [0,0.5] \\
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
\left[ 2\alpha(a_2 - a_1) + a_1, -2\alpha(a_5 - a_3) + a_4 \tilde{R}(b) \right] \\
\left[ 2\alpha(a_5 - a_3) + 2a_4 - 2a_2 - 2a_4 - a_1 \tilde{R}(b) \right] \\
\left[ 2\alpha(b_2 - b_1) + b_1, -2\alpha(b_5 - b_3) + b_3 \tilde{R}(b) \right] \\
\left[ 2\alpha(b_5 - b_3) + 2b_2 - 2b_5 - 2b_2 + b_1 \tilde{R}(b) \right] \end{array} \right. 
\text{ for } \alpha \in [0,0.5]
\end{align*}
\]
We have to prove \((A_f/J_B)_α = A_α(J/B)_α\).

First, we take,

\[ A_f(J/B)_α = \left( \frac{α_1}{K(b)'/R(b)'}, \frac{α_2}{K(b)'/R(b)'}, \frac{α_3}{K(b)'/R(b)'}, \frac{α_4}{K(b)'/R(b)'}, \frac{α_5}{K(b)'/R(b)'} \right) \]

Where \(K(b) = \frac{(b_1 + b_2 + b_3 + b_4 + b_5)}{α}\).

\[ (A_f(J/B)_α) = \left( \frac{2α (α_2 - α_4) + α_3}{K(b)'} - \frac{α_2 (α_2 - α_4) + α_3}{K(b)'}, \frac{2α (α_2 - α_4) + α_3}{K(b)'} - \frac{α_2 (α_2 - α_4) + α_3}{K(b)'}, \frac{2α (α_2 - α_4) + α_3}{K(b)'} - \frac{α_2 (α_2 - α_4) + α_3}{K(b)'}, \frac{2α (α_2 - α_4) + α_3}{K(b)'} - \frac{α_2 (α_2 - α_4) + α_3}{K(b)'}, \frac{2α (α_2 - α_4) + α_3}{K(b)'} - \frac{α_2 (α_2 - α_4) + α_3}{K(b)'} \right) \]

Next, we take,

\[ A_α(J/B)_α = \left( \frac{α_1}{K(b)'/R(b)'}, \frac{α_2}{K(b)'/R(b)'}, \frac{α_3}{K(b)'/R(b)'}, \frac{α_4}{K(b)'/R(b)'}, \frac{α_5}{K(b)'/R(b)'} \right) \]

Where \(K(b)' = \frac{2α (b_1 - b_2) - (b_3 - b_4) + b_5}{α}\) for \(α ∈ [0,0.5]\).

\[ \tilde{K}(b)' = \frac{2α (b_1 - b_2) - (b_3 - b_4) + b_5}{α} \] for \(α ∈ [0,0.5]\).

\[ A_α(J/B)_α = \left( \frac{2α (b_1 - b_2) - (b_3 - b_4) + b_5}{K(b)}, \frac{2α (b_1 - b_2) - (b_3 - b_4) + b_5}{K(b)} - \frac{α}{K(b)}, \frac{2α (b_1 - b_2) - (b_3 - b_4) + b_5}{K(b)} - \frac{α}{K(b)}, \frac{2α (b_1 - b_2) - (b_3 - b_4) + b_5}{K(b)} - \frac{α}{K(b)}, \frac{2α (b_1 - b_2) - (b_3 - b_4) + b_5}{K(b)} - \frac{α}{K(b)} \right) \]

\[
\text{Property 6.5} \quad \text{The join operation of the } \alpha \text{ – Cuts of the two PFNs are equal to the } \alpha \text{ – Cut of their join. (i.e.)} \quad (A_f(J/B)_α = A_α(J/)B_α) \]

\[
\text{Proof} \quad \text{Let } A_f = (a_1, a_2, a_3, a_4, a_5) \text{ and } B_f = (b_1, b_2, b_3, b_4, b_5) \text{ be two pentagonal fuzzy numbers. Then, the } \alpha \text{ – Cuts of two pentagonal fuzzy numbers } A_f \text{ and } B_f \text{ are} \]
Now, we have to prove \((A_p(\vee)\mathcal{B}_p)_\alpha = A_{\vee}\mathcal{B}_\alpha\).

\[
A_{\vee} = \begin{cases}
[2\alpha(a_2 - a_1) + a_1, -2\alpha(a_5 - a_4) + a_5] & \text{for } \alpha \in [0, 0.5] \\
[2\alpha(a_2 - a_1) + 2a_2 - a_1, -2\alpha(a_4 - a_2) + 2a_4 - a_2] & \text{for } \alpha \in [0.5, 1]
\end{cases}
\]

\[
B_{\vee} = \begin{cases}
[2\alpha(b_2 - b_1) + b_1, -2\alpha(b_5 - b_4) + b_5] & \text{for } \alpha \in [0, 0.5] \\
[2\alpha(b_2 - b_1) + 2b_2 - b_1, -2\alpha(b_4 - b_2) + 2b_4 - b_2] & \text{for } \alpha \in [0.5, 1]
\end{cases}
\]

Now, we have to prove \((A_p(\vee)\mathcal{B}_p)_\alpha = A_{\vee}\mathcal{B}_\alpha\).

\[
A_{\vee}(\mathcal{B}_p) = \begin{cases}
[2\alpha(a_2 - a_1) + a_1, -2\alpha(a_5 - a_4) + a_5] \\
\{2\alpha(a_2 - a_1) + 2a_2 - a_1, -2\alpha(a_4 - a_2) + 2a_4 - a_2\}^\vee
\end{cases}
\]

\[
[2\alpha(b_2 - b_1) + b_1, -2\alpha(b_5 - b_4) + b_5] \\
\{2\alpha(b_2 - b_1) + 2b_2 - b_1, -2\alpha(b_4 - b_2) + 2b_4 - b_2\}^\vee
\]

\[
= \begin{cases}
[2\alpha(a_2 - a_1) + a_1, -2\alpha(a_5 - a_4) + a_5] \\
\{2\alpha(a_2 - a_1) + 2a_2 - a_1, -2\alpha(a_4 - a_2) + 2a_4 - a_2\}^\vee
\end{cases}
\]

\[
\begin{align*}
&\{\max[2\alpha(a_2 - a_1) + a_1, 2\alpha(b_2 - b_1) + b_1],\} \\
&\{\max[-2\alpha(a_5 - a_4) + a_5, -2\alpha(b_5 - b_4) + b_5]\} \\
&\{\max[2\alpha(a_2 - a_1) + a_1 - 2a_2, 2\alpha(b_2 - b_1) - 2b_1 + 2b_4 - b_2],\}
\end{align*}
\]

\[
\Rightarrow (A_p(\vee)\mathcal{B}_p)_\alpha = A_{\vee}\mathcal{B}_\alpha
\]

Hence, we have, \((A_p(\vee)\mathcal{B}_p)_\alpha = A_{\vee}\mathcal{B}_\alpha\).

**Property 6.6**

The meet operation of the \(\alpha\) – Cuts of the two PFNs are equal to the \(\alpha\) – Cut of their meet. (i.e.), \((A_p(\land)\mathcal{B}_p)_\alpha = A_{\land}\mathcal{B}_\alpha\).

**Proof**

Let \(A_p = (a_2, a_3, a_4, a_5)\) and \(B_p = (b_2, b_3, b_4, b_5)\) be two pentagonal fuzzy numbers. Then, the \(\alpha\) – Cuts of two pentagonal fuzzy numbers \(A_p\) and \(B_p\) are

\[
A_{\land} = \begin{cases}
[2\alpha(a_2 - a_1) + a_1, -2\alpha(a_5 - a_4) + a_5] & \text{for } \alpha \in [0, 0.5] \\
[2\alpha(a_2 - a_1) + 2a_2 - a_1, -2\alpha(a_4 - a_2) + 2a_4 - a_2] & \text{for } \alpha \in [0.5, 1]
\end{cases}
\]

\[
B_{\land} = \begin{cases}
[2\alpha(b_2 - b_1) + b_1, -2\alpha(b_5 - b_4) + b_5] & \text{for } \alpha \in [0, 0.5] \\
[2\alpha(b_2 - b_1) + 2b_2 - b_1, -2\alpha(b_4 - b_2) + 2b_4 - b_2] & \text{for } \alpha \in [0.5, 1]
\end{cases}
\]

Now, we have to prove \((A_p(\land)\mathcal{B}_p)_\alpha = A_{\land}\mathcal{B}_\alpha\).

\[
A_{\land}(\mathcal{B}_p) = \begin{cases}
[2\alpha(a_2 - a_1) + a_1, -2\alpha(a_5 - a_4) + a_5] \\
\{2\alpha(a_2 - a_1) + 2a_2 - a_1, -2\alpha(a_4 - a_2) + 2a_4 - a_2\}^\land
\end{cases}
\]

\[
[2\alpha(b_2 - b_1) + b_1, -2\alpha(b_5 - b_4) + b_5] \\
\{2\alpha(b_2 - b_1) + 2b_2 - b_1, -2\alpha(b_4 - b_2) + 2b_4 - b_2\}^\land
\]

\[
= \begin{cases}
[2\alpha(a_2 - a_1) + a_1, -2\alpha(a_5 - a_4) + a_5] \\
\{2\alpha(b_2 - b_1) + b_1, -2\alpha(b_5 - b_4) + b_5\}^\land
\end{cases}
\]

\[
\begin{align*}
&\{2\alpha(a_2 - a_1) + a_1, -2\alpha(a_5 - a_4) + a_5\}^\land \\
&\{2\alpha(b_2 - b_1) + b_1, -2\alpha(b_5 - b_4) + b_5\}^\land
\end{align*}
\]

\[
\Rightarrow (A_p(\land)\mathcal{B}_p)_\alpha = A_{\land}\mathcal{B}_\alpha
\]
Hence, we have,

\[
\left( (A \wedge \bar{B})_\alpha \right)_\alpha = A \wedge \bar{B}
\]

7. CONCLUSION
In this paper, a new arithmetic operations on pentagonal fuzzy numbers using \( \alpha \)-cut operations have been studied. Further, some important properties have been proved using the new proposed arithmetic operations. This notion can be extended to some of the optimization problems in future.

REFERENCES