CHROMATIC NUMBER TO THE TRANSFORMATION\( (G^-^-^-^-^-) \) OF \( K_{1,n} \) AND \( K_{m,n} \)

B. Stephen John\(^1\) and S. Andrin Shahila\(^2\)
Department of Mathematics, Annai Velankanni College\(^{1,2}\), Tholayavattam, Tamilnadu.India-629157
E-mail: stephenjohn1963@gmail.com\(^1\); andrinshahila@gmail.com\(^2\)

Abstract: Let \( G = (V,E) \) be an undirected simple graph. The transformation graph \( G^-^-^-^-^- \) of \( G \) is a simple graph with vertex set \( V(G) \cup E(G) \) in which adjacency is defined as follows: (a) two elements in \( V(G) \) are adjacent if and only if they are non-adjacent in \( G \), (b) two elements in \( E(G) \) are adjacent if and only if they are non-adjacent in \( G \), and (c) an element of \( V(G) \) and an element of \( E(G) \) are adjacent if and only if they are non-incident in \( G \). In this paper, we determine the chromatic number of Transformation graph \( G^-^-^-^-^- \) for Star and Complete Bipartite graph.

Keywords: Star Graph, Complete Bipartite Graph, Chromatic Number, Transformation Graph

1. INTRODUCTION:

In this paper, we are concerned with finite, simple graph. Let \( G = (V,E) \) be a graph, if there is an edge \( e \) joining any two vertices \( u \) and \( v \) of \( G \), we say \( u \) and \( v \) are adjacent. An n-vertex colouring or an n-colouring of a graph \( G = (V,E) \) is a mapping

\[ f: V \rightarrow S, \text{ where } S \text{ is a set of } n \text{-colours}. \]

Definition: 1.1

A graph \( G \) is an ordered pair \( (V(G), E(G)) \) consisting of a non-empty set \( V(G) \) of vertices and a set \( E(G) \), disjoint from \( V(G) \) of edges together with an incidence function \( \psi_G \) that associates with each edge of \( G \) an unordered pair of vertices of \( G \).

Definition: 1.2

A colouring \( C \) of a simple graph \( G \) is proper if no two adjacent vertices are assigned the same colour.

A graph is properly coloured if it is coloured with the minimum possible number of colours.

Definition: 1.3

The chromatic number of a graph \( G \) is the minimum number of colours required to colour \( G \) and is denoted by \( \chi(G) \).

Definition: 1.4

The total graph \( T(G) \) of a graph \( G \) is the graph whose vertex set is \( V(G) \cup E(G) \) and two vertices are adjacent in \( T \) if and only if they are either adjacent or incident in \( G \).

Definition: 1.5

The complement \( \bar{G} \) of a graph \( G \), which has \( V(G) \) as its set of points and two points are adjacent in \( \bar{G} \) if and only if they are not adjacent in \( G \).
Definition: 1.6

A **bipartite graph** is a graph whose vertex set can be partitioned into two subsets \( V_1 \) and \( V_2 \), such that each edge has one end in \( V_1 \) and one end in \( V_2 \).

A bipartite graph is said to be complete if every vertex of \( V_1 \) is joined to every vertex of \( V_2 \). A complete bipartite graph with \( |V_1| = m \) and \( |V_2| = n \) is denoted by \( K_{m,n} \).

\( K_{1,n} \) is called as **star graph** for \( n \geq 1 \).

In [2] generalized the concept of total graphs to a transformation graph \( G^{xyz} \) with \( x, y, z; \{-, +\} \), where \( G^{+++} \) is the total graph of \( G \), and \( G^{---} \) is its complement. Also, \( G^{--+}, G^{+-+} \) and \( G^{++-} \) are the complement of \( G^{+++}, G^{+-+} \) and \( G^{++-} \) respectively.

Here we investigate the transformation graph \( G^{---} \) of some graphs.

**Lemma: 1.1** For any complete graph \( K_n \), \( \chi(K_n) = n \).

**Lemma: 1.2** The chromatic number of \( \overline{K_n} \) is 1 and \( \chi(K_{m,n}) = 2 \).

**Theorem: 2.1**

Let \( G \) be a star graph with \( n + 1 \) vertices, that is \( G = K_{1,n} \), then the chromatic number of \( G^{---} \) is \( n \), that is \( \chi(G^{---}) = n \).

**Proof:**

The proof of this theorem is on induction.

If \( n = 1 \), \( G = K_{1,1} \) and \( G^{---} \) are represented in figure.1 as below.

Now, \( G^{---} \) is a null graph with three vertices.

By lemma: 1.2, \( \chi(G^{---}) = 1 \).

Therefore, the result is true form \( n = 1 \).

If \( n = 2 \), \( G = K_{1,2} \) is represented in figure.2 as below.

The transformation of \( G \), that is \( G^{---} \) is represented in figure.3.
Let $V(G^{---})$ be the vertex set of $G^{---}$, $V(G^{---}) = \{u, v_1, v_2, e_1, e_2\}$. Now divide the vertex set into two sets $V_1$ and $V_2$ such that $V_1 = \{v_1, v_2\}$ and $V_2 = \{u, e_1, e_2\}$.

Clearly, the induced subgraph formed by the vertices of $V_1$ is a complete subgraph $K_2$ in $G^{---}$. Hence, we need $|V_1|$ colours to colour the vertices of $V_1$. Let it be $c_1$ and $c_2$. Also, the subgraph formed by the vertices of $V_2$ is an independent set in $G^{---}$ and $u$ is an isolated vertex. The vertex $e_i$ is non-adjacent to $v_i$ for all $(i = 1, 2)$, so we can give the colour $c_1$ to the vertex $e_1$ which was assigned to the vertex $v_1$ and the colour $c_2$ can be given to the vertex $e_2$ which was assigned to the vertex $v_2$. Hence, the connected subgraph of $G^{---}$ can be coloured by the colours $c_1, c_2$. Since, $u$ is an isolated vertex of $G^{---}$, we can colour $u$ either by $c_1$ or $c_2$. Therefore, $\chi(K_{1,2}^{---}) = 2$.

Hence, the result is true for $n = 2$.

Assume the result is true for $n - 1$, that is $\chi(K_{1,n-1}^{---}) = n - 1$ and to prove $\chi(K_{1,n}^{---}) = n$.

The graph $G = K_{1,n}$ is represented in figure 4 as below.

![Figure 4(G)](image)

The transformation $G^{---}$ is represented in figure 5.

Let $V(G^{---}) = \{v_i, u_i, e_i/i\}$ set of $G^{---}$. Divide the vertex set into two sets $V_1$ and $V_2$ such that $V_1 = \{v_i/i = 1, 2, \ldots, (n-1)\}$ and $V_2 = \{u, e_i/i = 1, 2, \ldots, (n-1)\}$.

Clearly, the induced subgraph formed by the vertices of $V_1$ is a complete subgraph $K_{n-1}$ of $G^{---}$. Hence, we need $|V_1|$ colours to colour the vertices of $V_1$. Let it be $c_1, c_2, \ldots, c_{n-1}$. The subgraph formed by the vertices of $V_2$ is an independent set in $G^{---}$ and $u$ is an isolated vertex. Also, the vertex $e_i$ is non-adjacent to $v_i$ for all $i = 1, 2, \ldots, (n-1)$. Hence the connected subgraph of $G^{---}$ is coloured by $(n-1)$-
colours say \( c_i (i=1,2,\ldots,(n-1)) \). Since \( u \) is an isolated vertex of \( G^{---} \), we can colour \( u \) by any one colour of \( c_i (i=1,2,\ldots,(n-1)) \). Now the vertices \( v_n \) and \( e_n \) which are adjacent with all other vertices except \( u \), also \( v_n \) and \( e_n \) are non-adjacent.

Therefore, \( V_1 \cup \{v_n\} \) form a complete subgraph with \( n \)-vertices. Since the vertices of \( V_1 \) is coloured by \( \{c_i (i=1,2,\ldots,(n-1))\} \) colours. (By induction), so we need another colour \( c_n \) to colour the vertices \( v_n \) and \( e_n \). Hence, \( \chi(G^{---}) = n \).

**Theorem: 2.2**

Let \( G \) be any complete bipartite graph with \( (n + m) \) vertices, that is \( G = K_{m,n} \). Then the chromatic number of \( G^{---} \) is \( n + m - 1 \), that is \( \chi(G^{---}) = n + m - 1 \).

**Proof:**

\[
\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure7}
\caption{\( G = K_{m,n} \)}
\end{figure}

\[
\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure8}
\caption{\( G = K_{m,n}^{---} \)}
\end{figure}

Let \( G = K_{m,n} \) and the vertex set of \( G \) is \( V(G) = \{v_i, u_j / i = 1,2,\ldots,m; j = 1,2,\ldots,n \} \)

Let \( V_1 = \{v_1, v_2,\ldots,v_m\} \) and \( V_2 = \{u_1, u_2,\ldots,u_n\} \) be the vertex set and \( e_{1,1}, e_{1,2},\ldots,e_{m,n} \) are the edges of \( G \).

Therefore, \( V(G^{---}) = \{v_i, u_j, e_{i,j} / i = 1,2,\ldots,m; j = 1,2,\ldots,n \} \).

Let \( c_i (i=1,2,\ldots) \) be the required colours to colour the vertices of \( G^{---} \).
Choose the vertex $v_1$ and it is coloured by the colour $c_1$. Clearly the vertices $e_{1,j}$ are non-adjacent with $v_1$, so we can use the colour $c_1$ to the vertex $e_{1,j}$. But the vertices $\{v_i/(i=2,3,..,m)\}$ which are adjacent to $v_1$ and the vertices $u_j’s$ are independent with the respective $\{e_{1,j}; (j=1,2,..,n)\}$, so the vertices $\{e_{1,j}; (j=1,2,..,n)\}$ can be coloured by the existing colour $c_1$. Since each vertex $\{e_{1,j}; (j=1,2,..,n)\}$ is adjacent with $v_2$, we use another new colour $c_2$ to the vertices $v_2$ and $\{e_{2,j}; (j=1,2,..,n)\}$.

Also, the vertices $\{v_i; i=3,4,..,m\}$ which are adjacent to both $v_1$ and $v_2$ and the vertices $u_j’s$ are independent with the respective $\{e_{2,j}; (j=1,2,..,n)\}$, so the vertices $\{e_{2,j}; (j=1,2,..,n)\}$ can be coloured by the existing colour $c_2$. Since each vertex $\{e_{2,j}; (j=1,2,..,n)\}$ is adjacent with $v_3$, we use another new colour $c_3$ to the vertices $v_3$ and $\{e_{3,j}; (j=1,2,..,n)\}$.

Repeat the above process until we have to colour the vertices $v_{m-1}$ and $\{e_{m-1,j}; (j=1,2,..,n)\}$. We need $(m-1)$ –colours to colour the vertex set

$$\{v_i, e_{i,j} / i = 1,2, \ldots, m; j = 1,2, \ldots, n\}.$$  

(1)

Now, the vertex $v_m$ is non-adjacent with the vertices $u_j$ and $\{e_{m,j}; (j=1,2,..,n)\}$ which is an induced subgraph of the form $K_{1,n}^{---}$ and the existing $(m-1)$-colours cannot be given to the induced subgraph $K_{1,n}^{---}$ of $G^{---}$, so we need another new $n$-colours to colour the vertices of $K_{1,n}^{---} \{v_m, e_{m,j}, u_j\}$ for all $j = 1,2, \ldots, n$.

(2)

From (1) and (2)

We need $(m + n - 1)$-colours to colour $G^{---}$.

Therefore, $\chi(G^{---}) = m + n - 1$.

Hence the proof.

REFERENCES:


