CONTRA FINE SG-CONTINUOUS MAPS IN FINE TOPOLOGICAL SPACE

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Abstract: Powar P. L. and Rajak K. have introduced fine-topological space which is a special case of generalized topological space. Aim of this paper is we introduced in fine sg open sets in fine topological space and also we studied about we investigate properties of contra-F-sg-continuous maps in fine topological spaces

Keywords: Fsg-open sets, Fsg-closed set, contra-Fine semi-continuous maps, contra-F-sg-continuous maps

AMS Mathematical Subject classification (2010): 54C08, 54C10

1. INTRODUCTION

Powar P. L. and Rajak K.[11] have investigated a special case of generalized topological space called fine topological space. Continuous functions are the most important and most researched points in the whole of the Mathematical Science. Many different forms of continuous functions have been introduced over the years Levine[9] initiated the study about g-closed sets and he generalized the concept of closedness. Following this, in 1987, Bhattacharyya and Lahiri [2] introduced the notion of semi-generalized closed sets in topological spaces by means of semi-open sets of Levine . In continuation of this work, In this paper we investigate properties of contra-F-sg-continuous maps in fine topological spaces.

2. PRELIMINARIES

Definition: 2.1 [10,11]
Let (X, τ) be a topological space we define, τ(Αα)=τα={Ga(≠X):Ga∩Aα ≠ υ, for Aα∈τ and Aα ≠X, υ for some α ∈ J , where J is the index set}. Now, define ττ ={ {υ,X}∪{τα} . The above collection ττ of subsets of X is called the fine collection of subsets of X and (X, τ, ττ) is said to be the fine topological space X and generated by the topology τ on X.

The element of ττ are called fine open sets in (X, τ, ττ) and the complement of fine open set is called fine closed sets and it is denoted by ττc

Example: 2.2 [10,11]
Consider a topological space X = {1, 2, 3} with the topology

τ ={X, υ, {1}} ≡ {X, υ, Αα} where Αα = {1}. In view of Definition 2.1 we have, τα =τ(Αα)= τ{1}={{1},{1,2},{1, 3}}
then the fine collection is ττ ={ {υ,X}∪{τα}={ {υ,X, {1},{1, 2,}{1, 3}}.

We quote some important properties of fine topological spaces.
Lemma: 2.3 [10,11]
Let (X, τ, ττ) be a fine space then arbitrary union of fine open set in X is fine-open in X. 

Lemma: 2.4 [10,11]
The intersection of two fine-open sets need not be a fine-open set as the following example shows.
Example: 2.5
Let X = \{1, 2, 3\} be a topological space with the topology
\[\tau = \{X, \emptyset, \{1\}, \{2\}, \{1, 2\}\}, \tau_f = \{X, \emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}.\] It is easy to see that, the above collection \(\tau_f\) is not a topology. Since, \(\{1, 3\} \cap \{2, 3\} = \{3\} \not\in \tau_f\). Hence, the collection of fine open sets in a fine space X does not form a topology on X, but it is a generalized topology on X.

Remark: 2.6 [10,11]
In view of Definition 2.1 of generalized topological space and above Lemmas 2.1 and 2.2 it is apparent that \((X, \tau, \tau_f)\) is a special case of generalized topological space. It may be noted specifically that the topological space plays a key role while defining the fine space as it is based on the topology of X but there is no topology in the back of generalized topological space.

Definition 2.7 [10,11]
A subset A of a Fine space \((X, \tau, \tau_f)\) is said to be
(i) Fine semi-open if \(A \subseteq Fcl(Fint(A))\).
(ii) Fine pre open if \(A \subseteq Fint(Fcl(A))\).
(iii) Fine regular open if \(A = Fint(Fcl(A))\).

The complement of the above mentioned Fine open sets are called their respective Fine closed sets.

Definition 2.8
A subset A of a Fine space \((X, \tau, \tau_f)\) is called Fsg-closed if \(FscI(A) \subseteq U\) whenever \(A \subseteq U\) and U is Fine semi-open in Fine space \((X, \tau, \tau_f)\). The complement of Fsg-closed set is called Fsg-open.

Definition 2.9
The union of all Fsg-open sets, each contained in a set A in a Fine space \((X, \tau, \tau_f)\) is called the Fsg-interior of A and is denoted by Fsg-int(A).

Definition 2.10
The intersection of all Fsg-closed sets containing a set A in a space \((X, \tau, \tau_f)\) is called the Fsg-closure of A and is denoted by Fsg-cl(A).

Definition 2.11 [10,11]
A map \(f : (X, \tau, \tau_f) \rightarrow (Y, \sigma, \sigma_f)\) is called:
(i) Fine contra-continuous if \(f^{-1}(V)\) is Fine closed in Fine space X for each Fine open set V in Fine space Y;
(ii) contra Fine semi-continuous if \(f^{-1}(V)\) is Fine semi-closed in Fine space X for each Fine open set V in Fine space Y;
(iii) contra F-sg-continuous if \(f^{-1}(V)\) is F-sg-closed in Fine space X for each Fine open set V in Fine space Y;
(iv) F-sg-continuous if \(f^{-1}(V)\) is F-sg-closed in Fine space X for each Fine closed set V in Fine space Y;
(v) F-sg-irresolute if \(f^{-1}(V)\) is F-sg-closed in Fine space X for each F-sg-closed set V in Fine space Y;
(vi) Fine semi irresolute if \(f^{-1}(V)\) is Fine semi-closed in Fine space X for each Fine semi-closed set V in Fine space Y.

Definition 2.12
A Fine topological space \((X, \tau, \tau_f)\) is called
(i) a locally indiscrete if each Fine open subset of Fine space X is Fine closed in Fine space X;
(ii) Fine semi-T Schafer space if each F-sg-closed subset of Fine space X is Fine semi-closed in Fine space X;
(iii) F-sg-connected if Fine space X cannot be written as a disjoint union of two non-empty F-sg-open sets;
(iv) ultra normal if each pair of non-empty disjoint Fine closed sets can be separated by disjoint Fine clopen sets;
(v) weakly Hausdorff if each element of Fine space X is an intersection of Fine regular closed sets;
(vi) ultra Hausdorff if for each pair of distinct points \( x \) and \( y \) in Fine space \( X \), there exist Fine clopen sets \( A \) and \( B \) containing \( x \) and \( y \), respectively, such that \( A \cap B = \emptyset \).

**Result: 2.13**

Let \((X, \tau, \tau_f)\) be a Fine topological space. Then

(i) Every Fine semi-closed set of Fine space \( X \) is F-sg-closed in Fine space \( X \),
(ii) Every Fine closed set of Fine space \( X \) is F-sg-closed in Fine space \( X \),

Let \( S \) be a subset of a Fine space \( X \). The set \( \cap \{ U \in \tau : S \subseteq U \} \) is called the kernel of \( S \) and is denoted by \( \ker(S) \).

**Lemma: 2.14**

The following properties hold for the subsets \( U, V \) of a Fine topological space \((X, \tau, \tau_f)\)

(i) \( x \in \ker(U) \) if and only if \( U \cap G \neq \emptyset \) for any Fine closed set \( G \) containing \( x \).
(ii) \( U \subseteq \ker(U) \) and \( U = \ker(U) \) if \( U \) is Fine open in Fine space \( X \).
(iii) \( U \subseteq V \), then \( \ker(U) \subseteq \ker(V) \).

3. **CHARACTERIZATIONS OF CONTRA FINE SG-CONTINUOUS MAPS**

In this section we introduced contra F-sg-continuous and studied its characterization and investigate some of the basic properties.

**Definition 3.1**

A map \( f : (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f) \) is called contra F-sg-continuous if \( f^{-1}(V) \) is F-sg-closed in Fine space \( X \) for each Fine open set \( V \) in Fine space \( Y \);

**Remark:**

From the definitions we stated above, we observe that

(i) Every contra-Fine continuous map is contra F-sg-continuous.
(ii) Every contra Fine semi-continuous map is contra F-sg-continuous.

**Example : 3.2**

Let \( X = Y = \{a, b, c\} \), \( \tau = \{\emptyset, X, \{b\}\} \), \( \tau_f = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\} \) and \( \sigma = \{\emptyset, Y, \{c\}\} \). \( \sigma_f = \{\emptyset, Y, \{c\}, \{a, c\}, \{b, c\}\} \)

Define \( f : (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f) \) as \( f(a) = c; f(b) = a; f(c) = b \). Clearly \( f \) is contra F-sg-continuous map

**Example : 3.3**

Let \( X = Y = \{a, b, c\} \), \( \tau = \{\emptyset, X, \{a\}, \{b, c\}\} \), \( \tau_f = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\} \) and \( \sigma = \{\emptyset, Y, \{a\}\} \). \( \sigma_f = \{\emptyset, Y, \{a\}, \{a, b\}, \{a, c\}\} \)

Define \( f : (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f) \) as \( f(a) = b; f(b) = c; f(c) = a \).

Clearly \( f \) is contra F-sg-continuous map

**Theorem: 3.4**

Let \( f : (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f) \) be a map. The following statements are equivalent.

(i) \( f \) is contra F-sg-continuous.
(ii) The inverse image of each Fine closed set in Fine space \( Y \) is F-sg-open in Fine space \( X \).

**Proof**

Let \( G \) be a Fine closed set in Fine space \( X \). Then \( Y \setminus G \) is an Fine open set in Fine space \( Y \). By the assumption of (i), \( f^{-1}(Y \setminus G) = X \setminus f^{-1}(G) \) is F-sg-closed in Fine space \( X \). It implies that \( f^{-1}(G) \) is F-sg-open in Fine space \( X \). Converse is similar.

**Theorem: 3.5**

Suppose that F-SGC(X) is closed under arbitrary intersections. Then the following are equivalent for a map

\( f : (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f) \)

(i) \( f \) is contra F-sg-continuous.
(ii) the inverse image of every Fine closed set of Fine space \( Y \) is F-sg-open in Fine space \( X \).
Theorem: 3.10

For each \( x \in X \) and each Fine closed set \( B \) in Fine space \( Y \) with \( f(x) \in B \), there exists an F-sg-open set \( A \) in Fine space \( X \) such that \( x \in A \) and \( f(A) \subseteq B \).

(iv) \( f(F_{sgcl}(A)) \subseteq \ker(f(A)) \) for every subset \( A \) of Fine space \( X \).

(v) \( F_{sgcl}(f^{-1}(B)) \subseteq f^{-1}(\ker B) \) for every subset \( B \) of Fine space \( Y \).

Proof

(i) \( \supseteq \) (iii): Let \( x \in X \) and \( B \) be a Fine closed set in Fine space \( Y \) with \( f(x) \in B \). By (i), it follows that \( f^{-1}(Y \setminus B) = X \setminus f^{-1}(B) \) is F-sg-closed and so \( f^{-1}(B) \) is F-sg-open. Take \( A = f^{-1}(B) \). We obtain that \( x \in A \) and \( f(A) \subseteq B \).

(iii) \( \supseteq \) (ii): Let \( B \) be a Fine closed set in Fine space \( Y \) with \( x \in f^{-1}(B) \). Since \( f(x) \in B \), by (iii) there exists an F-sg-open set \( A \) in Fine space \( X \) containing \( x \) such that \( f(A) \subseteq B \). It follows that \( x \in A \subseteq f^{-1}(B) \). Hence \( f^{-1}(B) \) is F-sg-open.

(ii) \( \supseteq \) (i): Follows from the previous theorem.

(iii) \( \supseteq \) (iv): Let \( A \) be any subset of Fine space \( X \). Let \( y \not\in \ker(f(A)) \). Then there exists a Fine closed set \( F \) containing \( y \) such that \( f(A) \cap F = \emptyset \). Hence, we have \( A \cap f^{-1}(F) = \emptyset \) and \( F_{sgcl}(A) \cap f^{-1}(F) = \emptyset \). Hence we obtain \( f(F_{sgcl}(A)) \cap F = \emptyset \) and \( y \not\in f(F_{sgcl}(A)) \). Thus, \( f(F_{sgcl}(A)) \subseteq \ker(f(A)) \).

(iv) \( \supseteq \) (v): Let \( B \) be any subset of Fine space \( Y \). By (iv), \( f(F_{sgcl}(f^{-1}(B))) \subseteq \ker(B) \) and \( F_{sgcl}(f^{-1}(B)) \subseteq f^{-1}(\ker(B)) \).

(v) \( \supseteq \) (i): Let \( B \) be any Fine open set of Fine space \( Y \). By (v), \( F_{sgcl}(f^{-1}(B)) \subseteq f^{-1}(\ker(B)) \) and \( f_{sgcl}(f^{-1}(B)) = f^{-1}(B) \) and \( F_{sgcl}(f^{-1}(B)) = f^{-1}(B) \).

We obtain that \( f^{-1}(B) \) is F-sg-closed in Fine space \( X \).

Definition: 3.6

The graph \( G(f) \) of a map \( f : (X, \tau, \tau_f) \rightarrow (Y, \sigma, \sigma_f) \) is said to be contra F-sg-graph if for each \( (x, y) \in (X \times Y) \setminus G(f) \), there exist an F-sg-open set \( U \) in Fine space \( X \) containing \( x \) and a closed set \( V \) in Fine space \( Y \) containing \( y \) such that \( (U \times V) \cap G(f) = \emptyset \).

Definition: 3.7

A Fine topological space \( (X, \tau, \tau_f) \) is said to be locally F-sg-indiscrete if every F-sg-open set of Fine space \( X \) is Fine closed in Fine space \( X \).

Theorem: 3.8

If \( f : (X, \tau, \tau_f) \rightarrow (Y, \sigma, \sigma_f) \) is contra F-sg-continuous with Fine space \( X \) as locally F-sg-indiscrete, then \( f \) is Fine continuous.

Proof

Omitted.

Theorem: 3.9

Suppose that \( (X, \tau, \tau_f) \) and \( (Y, \sigma, \sigma_f) \) are Fine spaces and F-SGO(X) is Fine closed under arbitrary unions. If a map \( f : (X, \tau, \tau_f) \rightarrow (Y, \sigma, \sigma_f) \) is contra F-sg-continuous and \( Y \) is Fine regular, then \( f \) is F-sg-continuous.

Proof

Let \( x \) be an arbitrary point of Fine space \( X \) and \( V \) be an Fine open set of Fine space \( Y \) containing \( f(x) \). Since Fine space \( Y \) is Fine regular, there exists an Fine open set \( G \) in Fine space \( Y \) containing \( f(x) \) such that \( cl(G) \subseteq V \). Since \( f \) is contra F-sg-continuous, there exists \( U \in F\text{-SGO}(X) \) containing \( x \) such that \( f(U) \subseteq F cl(G) \). Then \( f(U) \subseteq F cl(G) \subseteq V \). Hence \( f \) is F-sg-continuous.

Theorem: 3.10

A contra F-sg-continuous image of a F-sg-connected space is connected.

Proof

Let \( f : (X, \tau, \tau_f) \rightarrow (Y, \sigma, \sigma_f) \) be a contra F-sg-continuous map of a F-sg-connected Fine space \( X \) onto a Fine topological space \( Y \). If possible, let Fine space \( Y \) be disconnected. Let \( A \) and \( B \) form a disconnection of Fine space \( Y \). Then \( A \) and \( B \) are Fine clopen and \( Y = A \cup B \) where \( A \cap B = \emptyset \). Since \( f \) is a
contra F-sg-continuous map. Fine space $X = f^1(Y) = f^1(A \cup B) = f^1(A) \cup f^1(B)$, where $f^1(A)$ and $f^1(B)$ are non-empty F-sg-open sets in Fine space $X$. Also $f^1(A) \cap f^1(B) = \emptyset$. Hence Fine space $X$ is not F-sg-connected. This is a contradiction. Therefore Fine space $Y$ is connected.

**Theorem: 3.11**

Let Fine space $(X, \tau, \tau_f)$ be F-sg-connected. Then each contra F-sg-continuous map of Fine space $X$ into a discrete Fine space $(Y, \sigma, \sigma_\tau)$ with at least two points is a constant map.

**Proof**

Let $f : (X, \tau, \tau_f) \rightarrow (Y, \sigma, \sigma_\tau)$ be a contra F-sg-continuous map. Then Fine space $X$ is covered by F-sg-open and F-sg-closed covering $\{f^1({y}) : y \in Y\}$. By assumption $f^1({y}) = \emptyset$ or Fine space $X$ for each $y \in Y$. If $f^1({y}) = \emptyset$ for all $y \in Y$, then $f$ fails to be a map. Then there exists only one point $y \in Y$ such that $f^1({y}) \neq \emptyset$ and hence $f^1({y}) = X$ which shows that $f$ is a constant map.

**Theorem: 3.12**

If $f$ is a contra F-sg-continuous map from a F-sg-connected Fine space $(X, \tau, \tau_f)$ onto any Fine space $(Y, \sigma, \sigma_\tau)$, then Fine space $(Y, \sigma, \sigma_\tau)$ is not a discrete space.

**Proof**

Suppose that Fine space $(Y, \sigma, \sigma_\tau)$ is discrete. Let $A$ be a proper nonempty Fine open and Fine closed subset of Fine space $Y$. Then $f^1(A)$ is a proper nonempty F-sg-open and F-sg-closed subset of Fine space $X$, which is a contradiction to the fact that Fine space $X$ is F-sg-connected.

**Definition: 3.13**

A Fine space $(X, \tau, \tau_f)$ is said to be F-sg-normal if each pair of non-empty disjoint Fine closed sets can be separated by disjoint F-sg-open sets.

**Theorem: 3.14**

If $f : (X, \tau, \tau_f) \rightarrow (Y, \sigma, \sigma_f)$ is a contra F-sg-continuous, Fine closed, injection and Fine space $Y$ is ultra normal, then Fine space $X$ is F-sg-normal.

**Proof**

Let $F_1$ and $F_2$ be disjoint Fine closed subsets of Fine space $(X, \tau, \tau_f)$. Since $f$ is Fine closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint Fine closed subsets of Fine space $Y$. Since Fine space $Y$ is ultra normal, $f(F_1)$ and $f(F_2)$ are separated by disjoint Fine clopen sets $V_1$ and $V_2$ respectively. Hence $F_i \subseteq f^1(V_i)$, $f^1(V_i)$ is F-sg-open in Fine space $X$ for $i = 1, 2$ and $f^1(V_1) \cap f^1(V_2) = \emptyset$. Thus, Fine space $X$ is F-sg-normal.

4. COMPOSITION OF MAPS

**Theorem: 4.1**

The composition of two contra F-sg-continuous maps need not be contra F-sg-continuous.

The following example supports the above theorem.

**Example: 4.2**

Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\sigma_f = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, X\}$

$\sigma = \{\emptyset, Y, \{b, c\}\}$ $\sigma_\tau = \{\emptyset, \{b\}, \{c\}, \{b,c\}, \{a,b\}, \{a,c\}, Y\}$. And $\rho = \{\emptyset, Z, \{c\}\}$ $\rho_\tau = \{\emptyset, \{c\}, \{b,c\}, \{a,c\}, Z\}$

Then the identity map $f : (X, \square, \square_\square) \rightarrow (Y, \sigma, \sigma_\tau)$ is contra F-sg-continuous and the identity map $g : (Y, \sigma, \sigma_\tau) \rightarrow (Z, \rho, \rho_\tau)$ is contra F-sg-continuous. But their composition $g \circ f : (X, \square, \square_\square) \rightarrow (Z, \rho, \rho_\tau)$ is not contra F-sg-continuous.

**Theorem: 4.3**

Let Fine space $X$ and Fine space $Z$ be any Fine topological spaces and Fine space $Y$ be a semi-$T_{1\over 2}$ space. Let

$f : (X, \square, \square_\square) \rightarrow (Y, \sigma, \sigma_\tau)$ be an irresolute map and $g : (Y, \sigma, \sigma_\tau) \rightarrow (Z, \rho, \rho_\tau)$ be a contra F-sg-continuous map. Then

$g \circ f : (X, \square, \square_\square) \rightarrow (Z, \rho, \rho_\tau)$ is contra Fine semi-continuous map.

**Proof**

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Let $U$ be any Fine open set in Fine space $Z$. Since $g$ is contra $F$-sg-continuous, $g^{-1}(U)$ is F-sg-closed in Fine space $Y$. But Fine space $Y$ is Fine semi-$T_{1\frac{1}{2}}$ space. Therefore $g^{-1}(U)$ is Fine semi-closed in Fine space $Y$. Since $f$ is irresolute, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is Fine semi-closed in Fine space $X$. Thus, $g \circ f$ is contra Fine semi-continuous.

**Theorem 4.4**

Let $f : (X, \mathcal{G}, \mathcal{T}) \rightarrow (Y, \sigma, \tau)$ be F-sg- irresolute map and $g : (Y, \sigma, \sigma_l) \rightarrow (Z, \rho, \rho_l)$ be contra F-sg-continuous map. Then $g \circ f : (X, \mathcal{G}, \mathcal{T}) \rightarrow (Z, \rho, \rho_l)$ is contra F-sg-continuous.

**Proof**

Let $U$ be an Fine open set in Fine space $Z$. Then $g^{-1}(U)$ is F-sg-closed in Fine space $Y$ because $g$ is contra F-sg-continuous. Since $f$ is F-sg- irresolute, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is F-sg-closed in Fine space $X$. Therefore $g \circ f$ is contra F-sg-continuous.

**Corollary 4.5**

Let $f : (X, \mathcal{G}, \mathcal{T}) \rightarrow (Y, \sigma, \sigma_l)$ be F-sg- irresolute map and $g : (Y, \sigma, \sigma_l) \rightarrow (Z, \rho, \rho_l)$ be contra F-sg-continuous map. Then $g \circ f : (X, \mathcal{G}, \mathcal{T}) \rightarrow (Z, \rho, \rho_l)$ is contra F-sg-continuous.

**Definition 4.6**

A map $f : (X, \mathcal{G}, \mathcal{T}) \rightarrow (Y, \sigma, \sigma_l)$ is said to be pre F-sg-open if the image of every F-sg-open subset of Fine space $X$ is F-sg-open in Fine space $Y$.

**Theorem 4.7**

Let $f : (X, \mathcal{G}, \mathcal{T}) \rightarrow (Y, \sigma, \sigma_l)$ be surjective F-sg- irresolute pre F-sg-open and $g : (Y, \sigma, \sigma_l) \rightarrow (Z, \rho, \rho_l)$ be any map. Then $g \circ f : (X, \mathcal{G}, \mathcal{T}) \rightarrow (Z, \rho, \rho_l)$ is contra F-sg-continuous if and only if $g$ is contra F-sg-continuous.

**Proof**

The ‘if’ part is easy to prove. To prove the ‘only if’ part, let $g \circ f : (X, \mathcal{G}, \mathcal{T}) \rightarrow (Z, \rho, \rho_l)$ be contra F-sg-continuous and let $U$ be a Fine closed subset of Fine space $Z$. Then $(g \circ f)^{-1}(U)$ is a F-sg-open subset of Fine space $X$. That is $f^{-1}(g^{-1}(U))$ is F-sg-open. Since $f$ is F- pre sg-open, $f^{-1}(g^{-1}(U))$ is a F-sg-open subset of Fine space $Y$. So, $g^{-1}(U)$ is F-sg-open in Fine space $Y$. Hence $g$ is contra F-sg-continuous.

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