DIRECT PRODUCT OF SP-ALGEBRA

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Abstract: In this paper, properties of SP-Algebra have been investigated and direct product of SP Algebra has been introduced and some theorems were stated and proved.

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1. INTRODUCTION

Two aspect of abstract algebra were introduced in [1, 2, 3, 4]. Following this, many authors established various algebras which were acted as a generalization or subclasses of these two algebras. In all these papers, they investigated the algebraic structures in different algebras and made a comparison between these algebras. In 2016, K. Shanmuga Priya and M. Mullai established a new class of algebra known as SP-Algebra[5]. They used the concept of SP-Ideal in SP-Algebra to construct quotient SP-Algebra. Homomorphism and kernel of SP-Algebra were also defined by them. They extended SP-Algebra to SP-Ring[6] and discussed integral domain, Euclidean domain, prime element and unique factorization theorems. This SP-Ring concept is directly applied to polynomials named SP-Polynomials and using that Gauss Lemma and Eisenstein Criterion were established in [7].

In this paper, the direct product of SP-Algebra is introduced and some of its properties are discussed briefly.

2. DIRECT PRODUCT OF SP-ALGEBRA

Definition 2.1 [5] An Algebra \((X, *, e)\) of type \((2, 0)\) is said to be SP-Algebra if
\((SP_1): x*x = e,\)
\((SP_2): x * e = x,\)
\((SP_3): if x * y = e and y * x = e, then x = y,\)
\((SP_4): (x * y) * (y * z) = (x * z),\)
where \(x, y, z \in X\) and "*" is any binary operation and 'e' is any constant.

Example 2.2 [5] If \(X\) is any non-empty finite set and \(P(X)\) denotes the power set of \(X\), then \((P(X), \Delta, \phi)\) is a SP-Algebra, where \(\Delta = (A-B) \cup (B-A)\).

Definition 2.3 [5] A non-empty subset \(I\) of SP-Algebra \(X\) is said to be SP-Ideal if
1. \(e \in I,\)
2. for every \(x, y \in I, x * y \in I,\)
3. if \(y * x \in I\) and \(x \in I\), then \(y \in I, x, y \in X.\)

Example 2.4 [5] \(I = \{\phi, \{a\}, \{b\}, \{a, b\}\}\) is a SP-Ideal of \((P(X), \Delta, \phi)\), where \(X = \{a, b, c\}.\)
Definition 2.5 [5] Let X be any SP-Algebra and I be a SP-Ideal of X. Define
\[ I_x = \{ y \in X/ x \ast y \in I \} \text{, where } x \in I. \]

Definition 2.6 [5] Let X be any SP-Algebra and I be a SP-Ideal of X. Define
\[ X/I = \{ I_x/ x \in X \} \text{ and } I_x \ast I_y = I_{x+y} \text{ is well-defined. This X/I is called Quotient SP-Algebra.} \]

Definition 2.7 [5] A map \( f \) from SP-Algebra \((X, \ast, e)\) into SP-Algebra \((X', \Delta, e')\) is said to be homomorphism if \( f(x \ast y) = f(x) \Delta f(y) \), where \( x, y \in X \).

Definition 2.8 Let \( X = (X, \ast, e_X) \) and \( Y = (Y, \Delta, e_Y) \) be two SP-Algebras.
The Direct product of X and Y be defined as \( X \times Y = (X \times Y, \delta, (e_X, e_Y)) \), where \( X \times Y = \{ (x, y)/ x \in X, y \in Y \} \), \((e_X, e_Y)\) is the constant element and \( \delta \) be the binary operation defined by \( (x_1, y_1, \delta (x_2, y_2) = (x_1 \ast x_2, y_1 \Delta y_2) \)

Theorem 2.9 The Direct product of two SP-Algebras is a SP-Algebra.
Proof:
Let \((X, \ast, e_X)\) and \(Y = (Y, \Delta, e_Y)\) be two SP-Algebras.
Then \( X \times Y \) is defined as \( X \times Y = \{(x, y)/ (x \in X, y \in Y)\} \).
Since \( e_X \in X \) and \( e_Y \in Y \), then \( (e_X, e_Y) \in X \times Y \).
Therefore, \( X \times Y \) is non-empty.
To prove \( X \times Y \) is SP-Algebra.
Let \((a, b), (c, d) \in X \times Y \).
1) \((a, b) \delta (a, b) = (a \ast a, b \Delta b) = (e_X, e_Y)\).
2) \((a, b) \delta (e_X, e_Y) = (a \ast e_X, b \Delta e_Y) = (a, b)\).
3) If \((a_1, b_1), (a_2, b_2) \in (e_X, e_Y)\) and \((a_2, b_2) \delta (a_1, b_1) = (e_X, e_Y)\),
\( \Rightarrow (a_1 \ast a_2, b_1 \Delta b_2) = (e_X, e_Y) \)
\( \Rightarrow a_1 \ast a_2 = e_X \) and \( b_1 \Delta b_2 = e_Y \)
\( \Rightarrow a_1 = a_2 \) and \( b_1 = b_2 \) \((by SP_4)\).
4) \( [(a_1, b_1) \delta (a_2, b_2)] \delta [(a_2, b_2) \delta (a_3, b_3) = [(a_1 \ast a_2, b_1 \Delta b_2) \delta (a_2 \ast a_3, b_2 \Delta b_3)] \)
\( = (((a_1 \ast a_2) \ast (a_2 \ast a_3)), (b_1 \Delta b_2) \ast (b_2 \Delta b_3))] \)
\( = (a_1 \ast a_3, b_1 \Delta b_2) \) \((by SP_4)\)
\( = (a_1, b_1) \delta (a_3, b_3) \)
Hence \( X \times Y \) is a SP-Algebra.

Corollary 2.10 If \( X_1, X_2, ..., X_n \) is a set of SP-Algebras, then \( X_1 \times X_2 \times ... \times X_n \) is also
SP-Algebra.

Theorem 2.11 Let \( X_1, X_2, ..., X_n \) and \( Y_1, Y_2, ..., Y_n \) are SP-Algebras and
let \( \phi_i : X_i \rightarrow Y_i, i = 1, 2, ..., n \) be a set of isomorphism.
If \( \phi : X_1 \times X_2 \times ... \times X_n \rightarrow Y_1 \times Y_2 \times ... \times Y_n \) given by
\( \phi(x_1, x_2, ..., x_n) = (\phi_1(x_1), \phi_2(x_2), ..., \phi_n(x_n)) \), then \( \phi \) is also an isomorphism.
Proof:
Let \( \phi_i : X_i \rightarrow Y_i, i = 1, 2, ..., n \) be a set of isomorphism.
Let \( \phi : X_1 \times X_2 \times ... \times X_n \rightarrow Y_1 \times Y_2 \times ... \times Y_n \) given by
\( \phi(x_1, x_2, ..., x_n) = (\phi_1(x_1), \phi_2(x_2), ..., \phi_n(x_n)) \).
To prove \( \phi \) is an isomorphism.
If \((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in X_1 \times X_2 \times ... \times X_n \), then
\( \phi[(x_1, x_2, ..., x_n) \delta (y_1, y_2, ..., y_n)] = \phi[x_1 \ast y_1, ... x_n \ast y_n] \)
\( = (\phi_1(x_1 \ast y_1), ..., \phi_n(x_n \ast y_n)) \)

Let $(y_1, y_2, ..., y_n) \in Y_1 \times Y_2 \times ... \times Y_n$

$\Rightarrow y_i \in Y_i$, there exists $x_i \in X_i$ such that $\phi_i(x_i) = y_i$ for $i = 1, 2, ..., n$, since $\phi_i$ is onto

$\Rightarrow (y_1, y_2, ..., y_n) = (\phi_1(x_1), \phi_2(x_2), ..., \phi_n(x_n)) = \phi(x_1, x_2, ..., x_n)$

$\Rightarrow \phi$ is onto.

To prove $\phi$ is 1-1,

$\phi(x_1, x_2, ..., x_n) = \phi(y_1, y_2, ..., y_n)$

$(\phi_1(x_1), \phi_2(x_2), ..., \phi_n(x_n)) = (\phi_1(y_1), \phi_2(y_2), ..., \phi_n(y_n))$

$\Rightarrow \phi_i(x_i) = \phi_i(y_i)$

$x_i = y_i$, where $i = 1, 2, ..., n$, since $\phi_i$ is 1-1

$\Rightarrow (x_1, x_2, ..., x_n) = (y_1, y_2, ..., y_n)$

$\Rightarrow \phi$ is 1-1

Hence $\phi$ is an isomorphism.

**Theorem 2.12** Let $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are SP-Algebras. If $\phi: X_1 \times X_2 \times ... \times X_n \rightarrow Y_1 \times Y_2 \times ... \times Y_n$ given by $\phi(x_1, x_2, ..., x_n) = (\phi_1(x_1), \phi_2(x_2), ..., \phi_n(x_n))$ is isomorphism, then each $\phi_i: X_i \rightarrow Y_i$, where $i = 1, 2, ..., n$ is also an isomorphism.

**Theorem 2.13** Let $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are SP-Algebras and let $\phi_i : X_i \rightarrow Y_i$, $i = 1, 2, ..., n$ and $\phi : X_1 \times X_2 \times ... \times X_n \rightarrow Y_1 \times Y_2 \times ... \times Y_n$ given by $\phi(x_1, x_2, ..., x_n) = \phi_1(x_1), \phi_2(x_2), ..., \phi_n(x_n)$ are homomorphisms, then ker$\phi = \ker \phi_1 \times \ker \phi_2 \times ... \times \ker \phi_n$.

**Proof:** Let $(x_1, x_2, ..., x_n) \in \ker \phi$

$\Rightarrow \phi(x_1, x_2, ..., x_n) = (e_{X_1}, e_{X_2}, ..., e_{X_n}) = (\phi_1(x_1), \phi_2(x_2), ..., \phi_n(x_n))$

$\Rightarrow \phi_i(x_i) = e_{X_i}$, for each $i = 1, 2, ..., n$.

$\Rightarrow (x_1, x_2, ..., x_n) \in \ker \phi_1 \times \ker \phi_2 \times ... \times \ker \phi_n$.

$\Rightarrow \ker \phi \subseteq \ker \phi_1 \times \ker \phi_2 \times ... \times \ker \phi_n$.

The converse follows similarly.

Therefore ker$\phi = \ker \phi_1 \times \ker \phi_2 \times ... \times \ker \phi_n$.

**Theorem 2.14** Let $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are SP-Algebras and let $\phi_i : X_i \rightarrow Y_i$, $i = 1, 2, ..., n$ and $\phi : X_1 \times X_2 \times ... \times X_n \rightarrow Y_1 \times Y_2 \times ... \times Y_n$ given by $\phi(x_1, x_2, ..., x_n) = (\phi_1(x_1), \phi_2(x_2), ..., \phi_n(x_n))$ are onto homomorphism, then $\phi(X_1 \times X_2 \times ... \times X_n) = \phi_1(X_1) \times \phi_2(X_2) \times ... \times \phi_n(X_n)$.

**Proof:** Let $(y_1, y_2, ..., y_n) \in \phi(X_1 \times X_2 \times ... \times X_n)$, there exists $(x_1, x_2, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$, such that $(y_1, y_2, ..., y_n) = \phi(x_1, x_2, ..., x_n)$, since $\phi$ is onto

$\Rightarrow (y_1, y_2, ..., y_n) = (\phi_1(x_1), \phi_2(x_2), ..., \phi_n(x_n))$

$\Rightarrow y_i = \phi_i(x_i)$

$\Rightarrow (y_1, y_2, ..., y_n) \in \phi_1(x_1) \times \phi_2(x_2) \times ... \times \phi_n(x_n)$

$\Rightarrow \phi(X_1 \times X_2 \times ... \times X_n) \subseteq \phi_1(X_1) \times \phi_2(X_2) \times ... \times \phi_n(X_n)$

Converse follows in a similar way.

Hence, $\phi(X_1 \times X_2 \times ... \times X_n) = \phi_1(X_1) \times \phi_2(X_2) \times ... \times \phi_n(X_n)$.

**Theorem 2.15** Let $(X_i, \ast, e_i)$ be a set of SP-Algebras and $I_i$ be a SP-Ideal of $X_i$, where $i = 1, 2, ..., n$, then $I_1 \times I_2 \times ... \times I_n$ is a SP-Ideal of $X_1 \times X_2 \times ... \times X_n$.
Proof:

Clearly, \((e_1, e_2, \ldots, e_n) \in I_1 \times I_2 \times \ldots \times I_n\), since \(e_i \in I_i\), \(i = 1, 2, \ldots, n\).
Let \((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in I_1 \times I_2 \times \ldots \times I_n\), where \(x_i, y_i \in I_i, i = 1, 2, \ldots, n\).
\(\Rightarrow x_i * y_i \in I_i, i = 1, 2, \ldots, n\), since each \(I_i\) is SP-Ideal.
\(\Rightarrow (x_1 * y_1, x_2 * y_2, \ldots, x_n * y_n) \in I_1 \times I_2 \times \ldots \times I_n\).
\(\Rightarrow (x_1, x_2, \ldots, x_n) \delta (y_1, y_2, \ldots, y_n) \in I_1 \times I_2 \times \ldots \times I_n\).
Let \((x_1, x_2, \ldots, x_n) \delta (y_1, y_2, \ldots, y_n)\) and \((x_1, x_2, \ldots, x_n) \in I_1 \times I_2 \times \ldots \times I_n\).
\(\Rightarrow x_i * y_i \in I_i, i = 1, 2, \ldots, n\).
\(\Rightarrow y_i \in I_i,\) since each \(I_i\) is SP-Ideal.
\(\Rightarrow (y_1, y_2, \ldots, y_n) \in I_1 \times I_2 \times \ldots \times I_n\).
Hence \(I_1 \times I_2 \times \ldots \times I_n\) is SP-Ideal.

Theorem 2.16
Let \((X_i, \ast, e_i)\) be a set of SP-Algebras and \(I_i\) be a SP-Ideal of \(X_i\), where \(i = 1, 2, \ldots, n\) and \(I_1 \times I_2 \times \ldots \times I_n\) is a SP-Ideal of \(X_1 \times X_2 \times \ldots \times X_n\),
then \((X_1 \times X_2 \times \ldots \times X_n)/(I_1 \times I_2 \times \ldots \times I_n)\) is isomorphic to \(((X_1/I_1) \times (X_2/I_2) \times \ldots \times (X_n/I_n))\).

Proof:
Let \(I = I_1 \times I_2 \times \ldots \times I_n\) and \(X = X_1 \times X_2 \times \ldots \times X_n\).
Define \(\phi: X/I \rightarrow ((X_1/I_1) \times (X_2/I_2) \times \ldots \times (X_n/I_n)),\)
given by \(\phi(I_{x_1,x_2,\ldots,x_n}) = (I_{x_1}, I_{x_2}, \ldots, I_{x_n})\).
Let \(I_{x_1,x_2,\ldots,x_n} = I_{y_1,y_2,\ldots,y_n}\).
\(\Rightarrow (x_1, x_2, \ldots, x_n) \sim (y_1, y_2, \ldots, y_n)\).
\(\Rightarrow (x_1, x_2, \ldots, x_n) \delta (y_1, y_2, \ldots, y_n) \in I_i,\) \(i = 1, 2, \ldots, n\).
\(\Rightarrow x_i \sim y_i,\) for all \(i = 1, 2, \ldots, n\).
\(\Rightarrow I_i(x_i) = I_i(y_i)\).
\(\Rightarrow (I_{1}, I_{2}, \ldots, I_{n})(x_n) = (I_{1}, I_{2}, \ldots, I_{n})(y_n)\).
\(\Rightarrow \phi(I_{x_1,x_2,\ldots,x_n}) = \phi(I_{y_1,y_2,\ldots,y_n})\).

Hence \(\phi\) is well-defined.
Let \(\phi(I_{x_1,x_2,\ldots,x_n}) = \phi(I_{y_1,y_2,\ldots,y_n})\).
\(\Rightarrow (I_{1}, I_{2}, \ldots, I_{n})(x_n) = (I_{1}, I_{2}, \ldots, I_{n})(y_n)\).
\(\Rightarrow I_{x_1,x_2,\ldots,x_n} = I_{y_1,y_2,\ldots,y_n}\).
\(\Rightarrow (x_1, x_2, \ldots, x_n) \sim (y_1, y_2, \ldots, y_n)\).
\(\Rightarrow x_i \sim y_i,\) for all \(i = 1, 2, \ldots, n\).
\(\Rightarrow I_{x_1,x_2,\ldots,x_n} = I_{y_1,y_2,\ldots,y_n}\).
\(\Rightarrow \phi\) is 1-1.
Let \((I_{1}, I_{2}, \ldots, I_{n})(x_n) \in ((X_1/I_1) \times (X_2/I_2) \times \ldots \times (X_n/I_n)).\)
\(\Rightarrow x_i \in X_i,\) for all \(i = 1, 2, \ldots, n\).
\(\Rightarrow (x_1, x_2, \ldots, x_n) \in X\).
\(\Rightarrow I_{x_1,x_2,\ldots,x_n} = I_{y_1,y_2,\ldots,y_n}\).
\(\Rightarrow \phi\) is onto.
Let \((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X/I\).
\(\Rightarrow \phi(I_{x_1,x_2,\ldots,x_n}) = \phi(I_{y_1,y_2,\ldots,y_n})\).
\(\Rightarrow x_i \sim y_i,\) for all \(i = 1, 2, \ldots, n\).
\(\Rightarrow \phi\) is isomorphic.
\[ = (I_{1x_1} \cdots I_{nx_n}) \delta (I_{1y_1} \cdots I_{ny_n}) \]
\[ = \phi(I_{1x_1}x_{2-}x_{n}) \delta \phi (I_{1y_1}y_{2-}y_{n}) \]

\[ \Rightarrow \phi \text{ is homomorphism} \]
\[ \Rightarrow \phi \text{ is isomorphism} \]

Hence \((X_1 \times X_2 \times \ldots \times X_n)/ (I_1 \times I_2 \times \ldots \times I_n) \) is isomorphic to \((X_1/I_1) \times (X_2/I_2) \times \ldots \times (X_n/I_n)\).

3 CONCLUSION

In this paper, Direct product of SP-Algebra is introduced. Also some theorems on Direct product of SP-Algebra are established here. In future, application of SP-Algebra and Rubik Cube problem will be investigated.

4. REFERENCES