OUTER-CLIQUE DOMINATION IN GRAPHS

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Abstract: Let G be a simple graph. A set S of vertices of a graph G is an outer-clique dominating set if every vertex not in S is adjacent to some vertex in S and the subgraph induced by V(G) \ S is clique. In this paper, we will show that given positive integers k, m and n such that 1 ≤ k ≤ m ≤ n − 1, there exists a connected nontrivial graph G with |V(G)| = n, \( \gamma(G) = k \), and \( \gamma_c(G) = m \). Further, we give characterization the outer-clique dominating sets resulting from the join of two graphs and give some important results. In this paper, we show that for each set of integers k, m, and n with 1 ≤ k ≤ m ≤ n − 1, the integers k, m, and n are realizable as domination number, outer-clique domination number, and order of G, respectively. Further, we give the characterization of the outer-clique dominating set with outer-clique domination numbers of 1 and 2. Finally, we characterize outer-clique dominating sets of the join of two graphs.

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1 INTRODUCTION

Let G be a simple graph. A subset S of a vertex set V(G) is a dominating set of G if for every vertex v ∈ V(G) \ S, there exists a vertex x ∈ S such that xv is an edge of G. The domination number \( \gamma(G) \) of G is the smallest cardinality of a dominating set \( \gamma(G) \). Dominating sets have several applications in a variety of fields, including communication and electrical networks, protection and location strategies, data structures and others. For more background on dominating sets, the reader may refer to [1, 2]. Dominination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [3].

A graph G is connected if there is at least one path that connects every two vertices \( x, y \in V(G) \), otherwise, G is disconnected. A set \( S \) of vertices of a graph G is an outer-connected dominating set if every vertex not in \( S \) is adjacent to some vertex in \( S \) and the sub-graph induced by \( V(G) \setminus S \) is connected. The outer-connected domination number \( \gamma_c(G) \) is the minimum cardinality of the outer-connected dominating set \( S \) of a graph G. The concept of outer-connected domination in graphs was introduced by Cyman [4].

For any two vertices u and v in a connected graph, the distance \( d_G(u,v) \) between u and v is the length of a shortest path in G. A u-v path of length \( d_G(u,v) \) is also referred to as u-v geodesic. The closed interval \( I_G[u,v] \) consist of all those vertices lying on a u-v geodesic in G. For a subset \( S \) of vertices of G, the union of all sets \( I_G[u,v] \) for \( u,v \in S \) is denoted by \( I_G[S] \). Hence \( x \in I_G[S] \) if and only if \( x \) lies on some u-v geodesic, where \( u,v \in S \). A set \( S \) is convex if \( I_G[S] = S \). Certainly, if G is a connected graph, then \( V(G) \) is convex. Convexity in graphs was studied in [5] while convex domination in graphs were found in the papers [6,7,8,9,10,11,12]. Concepts that are related to convex graph are complete graph and clique in graph.

A complete graph of order \( n \), denoted by \( K_n \), is the graph in which every pair of its distinct vertices are joined by an edge. A nonempty subset \( S \) of \( V(G) \) is a clique in G if the graph \( (S) \) induced by \( S \) is complete. A nonempty subset \( S \) of a vertex set \( V(G) \) is a clique dominating set of G if \( S \) is a dominating set and \( S \) is a clique in G. The minimum cardinality among all clique dominating sets of G, denoted by \( \gamma_c(G) \), is called the clique domination number of G. A clique dominating set \( S \) of G with \( |S| = \gamma_c(G) \) is called a \( \gamma_c \)-set of G. Clique dominating sets have a great diversity of applications. In setting up the communications links in a network one might want a strong core group that can communicate with each other member of the core group and so that everyone outside the core group could communicate with someone within the core group. A group of forest fire sentries that could see various sections of a forest might also be positioned in such a way that each could see the others in order to use triangulation to locate the site of a fire. In addition, the properties of dominating sets are useful in identifying structural properties of a social network [13,14].
Wolk[15] presents a forbidden subgraph characterization of a class of graphs which have a dominating clique of size one. He called such a dominating clique a central vertex or central point. The idea of Wolk was extended by Cozzens and Kelleher [16] to get forbidden subgraph conditions sufficient to imply the existence of a dominating set that induces a complete subgraph, a dominating clique. Daniel and Canoy[17] characterized the clique dominating sets in the join, corona, composition and Cartesian product of graphs and determine the corresponding clique domination number of the resulting graph. Other variants of clique domination in graphs are found in [18,19].

Motivated by the definition of outer-connected and clique domination in graphs, we define a new domination in graphs. A set \( S \) of vertices of a graph \( G \) is an outer-clique dominating set if every vertex not in \( S \) is adjacent to some vertex in \( S \) and the subgraph induced by \( V(G) \setminus S \) is clique. The outer-clique domination number of \( G \), denoted by \( \gamma_{cl}(G) \), is the minimum cardinality of an outer-clique dominating set of \( G \). A outer-clique dominating set of cardinality \( \gamma_{cl}(G) \) will be called a \( \gamma_{cl} - \)set.

In this paper, we will show that given positive integers \( k, m \) and \( n \) such that \( 1 \leq k \leq m \leq n - 1 \), there exists a connected nontrivial graph \( G \) with \( |V(G)| = n, \gamma(G) = k \), and \( \gamma_{cl}(G) = m \). Further, we give characterization the outer-clique dominating sets resulting from the join of two graphs and give some important results. For general concepts we refer the reader to [20].

2 RESULTS

In this paper, we denote by \( OC(G) \), a family of all graph \( G \) with outer-clique dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family \( OS(G) \).

From the definition of outer-clique dominating set, the following result is immediate.

Remark 2.1: Let \( G \) be a nontrivial connected graph of order \( n \). Then \( 1 \leq \gamma(G) \leq \gamma_{cl}(G) \leq n - 1 \).

It is worth mentioning that the upper bound in Remark 1 is sharp. For example, if \( G = K_1 + \bar{K} \), then \( \gamma_{cl}(G) = n - 1 \) for all \( n \geq 2 \). The lower bound is also attainable as the following result shows.

Theorem 2.2: Given positive integers \( k, m \) and \( n \) such that \( 1 \leq k \leq m \leq n - 1 \), there exists a connected nontrivial graph \( G \) with \( |V(G)| = n, \gamma(G) = k \), and \( \gamma_{cl}(G) = m \).

Proof: Consider the following cases:

Case 1. Suppose \( 1 < k = m < n - 1 \).

Let \( G \) be a join of a path (of order 1) and a complement of a complete graph (of order \( n - 1 \)), that is, \( G = P_1 + \bar{K}_{n-1} \), with \( n \geq 2 \). Then, the set \( V(P_1) \) is a \( k \)-set of \( G \), the set \( V(\bar{K}_{n-1}) \) is a \( \gamma_{cl} - \)set of \( G \). Thus, \( |V(G)| = 1 + (n - 1) = n, \gamma(G) = 1 = k \), and \( \gamma_{cl}(G) = n - 1 = m \) for all \( n \geq 2 \).

Case 2. Suppose \( 1 < k = m < n - 1 \).

Let \( G \) be a corona of a complete graph \( K_k = [x_1, x_2, \ldots, x_k] \) and a path \( P_1 \), that is, \( G = K_k \circ P_1 (k \geq 2) \). Let \( n = 2k \). Then the set \( V(K_k) \) is a \( k \)-set of \( G \) and the set \( \{V(P_1^1), V(P_1^2), \ldots, V(P_1^k)\} \) is a \( \gamma_{cl} - \)set of \( G \). Thus, \(|V(G)| = 2k = n, \gamma(G) = k \), and \( \gamma_{cl}(G) = k = m \) for all \( k \geq 2 \).

Case 3. Suppose \( 1 < k < m < n - 1 \).

Let \( G \) be a corona of a complete graph \( K_k = [x_1, x_2, \ldots, x_k] (k \geq 2) \), and a complement of a complete graph \( \bar{K}_r \) (\( r \geq 2 \)), that is, \( G = K_k \circ \bar{K}_r \). Let \( n = k + m \) and \( m = kr \). Then the set \( V(K_k) \) is a \( k \)-set of \( G \) and the set \( \bigcup_{v \in V(K_k)} \bar{K}_r \) is a \( \gamma_{cl} - \)set of \( G \). Thus, \(|V(G)| = k + kr = k + m = n, \gamma(G) = k \), and \( \gamma_{cl}(G) = kr = m \) for all \( k \geq 2 \) and \( r \geq 2 \). This proves the assertion.

The next result is an immediate consequence of Theorem 2.2.

Corollary 2.3: The difference \( \gamma_{cl}(G) - \gamma(G) \) can be made arbitrarily large.

Proof: Let \( n \) be a positive integer. By Theorem 2, there exists a connected graph \( G \) such that \( \gamma_{cl}(G) = n + 1 \) and \( \gamma(G) = 1 \). Thus, \( \gamma_{cl}(G) - \gamma(G) = n \), showing that \( \gamma_{cl}(G) - \gamma(G) \) can be made arbitrarily large.

The following remark is a quick observation of a parameter - outer-clique domination number.

Remark 2.4: If \( S \) is an outer-clique dominating set of \( G \), then \( S \) need not be clique and \( V(G) \setminus S \) need not be dominating.
For example, consider the graph $G$ below (see Figure 1). The set $S = \{p, x, w\}$ is an outer-clique dominating set in $G$ and $V(G) \setminus S = \{y, z, q\}$ is not a dominating set in $G$.

Figure 1: A graph $G$ with $\gamma_{cl}(G) = 3$.

The subsequent results presents the characterizations of an outer-clique domination number of one and an outer-clique domination number of two respectively.

**Theorem 2.5:** Let $G$ be a connected nontrivial graph. Then $\gamma_{cl}(G) = 1$ if and only if $G$ is a complete graph.

*Proof:* Suppose that $\gamma_{cl}(G) = 1$. Let $S = \{v\}$ be an $ap_{cl}$-set in $G$. Then $V(G) \setminus S$ is a clique in $G$. Suppose that there exists distinct vertices $x, y \in V(G) \setminus S$ such that $xy \not\in E(G)$. Since $S$ is a dominating set, $xv, yv \in E(G)$. Thus, $x, v, y \in I_G[x, y] \subseteq I_G[V(G) \setminus S]$. Since $v \not\in V(G) \setminus S$, it followsthat $I_G[V(G) \setminus S] = \emptyset$ contrary to our assumption that $V(G) \setminus S$ is a clique in $G$. Thus, for any $x, y \in V(G) \setminus S$, $xy \in E(G)$, that is, $(V(G) \setminus S)$ is complete. Hence, $G = \langle S \rangle + (V(G) \setminus S)$ is a complete graph.

For the converse, suppose that $G$ is a complete graph. Let $v \in V(G)$. Then $S = \{v\}$ is a dominating of $G$. Since $G$ is complete, $(V(G) \setminus S)$ is also complete, that is, $V(G) \setminus S$ is a clique in $G$. Accordingly, $S$ is an outer-clique dominating set in $G$. Hence, $\gamma_{cl}(G) = 1$. ■

**Theorem 2.6:** Let $G$ be a connected non-complete graph of order $n \geq 3$. Then $\gamma_{cl}(G) = 2$ if and only if there exists a dominating set $S = \{x, y\}$ such that $V(G) \setminus S$ is clique in $G$.

*Proof:* Suppose that $\gamma_{cl}(G) = 2$. Let $S = \{x, y\}$ be a $cl_{cl}$-set in $G$. Then $S$ is a dominating set and $V(G) \setminus S$ is clique in $G$.

For the converse, suppose that there exists a dominating set $S = \{x, y\}$ such that $V(G) \setminus S$ is clique in $G$. Then $S$ is an outer-clique dominating set of $G$. Thus, $\gamma_{cl}(G) \leq |S| = 2$. Since $G$ is non-complete, $\gamma_{cl}(G) \geq 2$. Therefore $\gamma_{cl}(G) = 2$. ■

The following result is a generalized form of Theorem 2.6.

**Theorem 2.7:** Let $G$ be a connected non-complete graph of order $n \geq 3$. Then $\gamma_{cl}(G) = k$ if and only if there exists a dominating set $S$ in $G$ such that $|S| = k$ and

$$|V(G) \setminus S| = \max\{|T| : T \text{ is clique in } G\} \text{ or } |V(G) \setminus S| = \max\{|T \setminus \{x\}| : T \text{ is clique in } G\}.$$

*Proof:* Suppose that $\gamma_{cl}(G) = k$. Let $S = \{x_1, \ldots, x_k\}$ be a $\gamma_{cl}$-set in $G$. Then $S$ is a dominating set and $V(G) \setminus S$ is clique in $G$. Thus, there exists a dominating set $S$ in $G$ such that $|S| = k$ and $|V(G) \setminus S| \leq \max\{|T| : T \text{ is clique in } G\}$ or $|V(G) \setminus S| \leq \max\{|T \setminus \{x\}| : T \text{ is clique in } G\}$.

Case 1. Consider $|V(G) \setminus S| \leq \max\{|T| : T \text{ is clique in } G\}$.

Since $S$ is a $ap_{cl}$-set in $G$, $|S| \leq |K|$ for all outer-clique $K$ in $G$. This implies that $T = V(G) \setminus K$ is clique in $G$ for all outer-clique $K$ in $G$. Thus, $|V(G) \setminus S| = |V(G)| - |S| \geq |V(G)| - |K| = |V(G) \setminus K| = |T|$, for all clique $T$ in $G$. This shows that $|V(G) \setminus S| \geq \max\{|T| : T \text{ is clique in } G\}$, that is, $|V(G) \setminus S| = \max\{|T| : T \text{ is clique in } G\}$.

Case 2. Consider $|V(G) \setminus S| \leq \max\{|T \setminus \{x\}| : T \text{ is clique in } G\}$.

Since $S$ is a $ap_{cl}$-set in $G$, $|S| \leq |K|$ for all outer-clique $K$ in $G$. This implies that for each $x \in T$, $T \setminus \{x\} = V(G) \setminus K$ is clique in $G$ for all outer-clique $K$ in $G$. Thus, $|V(G) \setminus S| = |V(G)| - |S| \geq |V(G)| - |K| = |V(G) \setminus K| = |T \setminus \{x\}|$, for all clique $T \setminus \{x\}$ in $G$. This shows that $|V(G) \setminus S| \geq \max\{|T \setminus \{x\}| : T \text{ is clique in } G\}$, that is, $|V(G) \setminus S| = \max\{|T \setminus \{x\}| : T \text{ is clique in } G\}$. 

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For the converse, suppose that there exists a dominating set \( S \) in \( G \) such that \( |S| = k \) and \( |V(G) \setminus S| = \max \{|T| : T \text{ is clique in } G\} \). Then \( S \) is an outer-clique dominating set of \( G \). Thus, \( \overline{\gamma}_c(G) \leq |S| = k \). Let \( S^0 \) be a \( \overline{\gamma}_c \)-set in \( G \). Then \( V(G) \setminus S^0 \) is a clique in \( G \). Let \( T^0 = V(G) \setminus S^0 \). Since \( |S^0| \leq |S| \) for all outer-clique \( S \) in \( G \), it follows that:

\[
\max \{|T| : T \text{ is clique in } G\} = |V(G) \setminus S| \quad \text{for all outer-clique } S \text{ in } G
\]

\[
|V(G) \setminus S^0| \quad \text{for all outer-clique } S^0 \text{ in } G
\]

\[
|T^0| \quad \text{for all clique } T^0 \text{ in } G
\]

\[
\leq \max \{|T| : T \text{ is clique in } G\}
\]

Therefore, \( |V(G) \setminus S| = |V(G) \setminus S^0| \), that is, \( |S| = |S^0| \). Therefore \( \overline{\gamma}_c(G) = |S^0| = |S| = k \). Similarly, if \( |V(G) \setminus S| = \max \{|T| : T \in G\} \), then \( \overline{\gamma}_c(G) = k \). \( \blacksquare \)

The join of two graphs \( G \) and \( H \) is the graph \( G + H \) with vertex-set \( V(G + H) = V(G) \cup V(H) \) and edge-set

\[
E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.
\]

The next result is the characterization of an outer-clique dominating set in the join of two graphs.

**Theorem 2.8:** Let \( G \) and \( H \) be nontrivial connected graphs. Then a nonempty subset \( S = S_G \cup S_H \) of \( V(G + H) \) is an outer-clique dominating set in \( G + H \) where \( S_G \subseteq V(G) \) and \( S_H \subseteq V(H) \) if and only if one of the following statements is satisfied:

(i) If \( G \) is a complete graph, then \( S_G = \emptyset \) or \( S_G = V(G) \setminus C \) for any nonempty set \( C \subseteq V(G) \), and \( S_H = V(H) \) or \( S_H = V(H) \setminus D \) is a dominating set of \( H \) where \( D \) is clique in \( H \).

(ii) If \( H \) is a complete graph, then \( S_H = \emptyset \) or \( S_H = V(H) \setminus C \) for any nonempty set \( C \subseteq V(H) \), and \( S_G = V(G) \) or \( S_G = V(G) \setminus D \) is a dominating set of \( G \) where \( D \) is clique in \( G \).

(iii) \( V(G) \setminus S_G \) is clique in \( G \) and \( V(H) \setminus S_H \) is clique in \( H \).

**Proof:** Suppose that a subset \( S = S_G \cup S_H \) of \( V(G + H) \) is an outer-clique dominating set in \( G + H \) where \( S_G \subseteq V(G) \) and \( S_H \subseteq V(H) \). Then \( S \) is a dominating set of \( G \) and \( V(G) \setminus S \) is clique in \( G \). Consider the following cases:

**Case 1.** Suppose that \( G \) is a complete graph. Let \( C \) be any nonempty subset of \( V(G) \). If \( V(G) \cap S = \emptyset \), then \( S_G = \emptyset \) and \( S_H = V(H) \) or \( S_H = V(H) \setminus D \) is a dominating set of \( H \) where \( D \) is clique in \( H \). If \( V(G) \cap S \neq \emptyset \), then \( S_G = V(G) \setminus C \) and \( S_H = V(H) \setminus S_G \setminus V(H) \setminus \) is a dominating set of \( H \) where \( D \) is clique in \( H \). This shows statement (i).

**Case 2.** Suppose that \( H \) is a complete graph. Let \( C \) be any nonempty subset of \( V(H) \). If \( V(H) \cap S = \emptyset \), then \( S_H = \emptyset \) and \( S_G = V(G) \setminus S_G \setminus V(G) \setminus \) is a dominating set of \( G \) where \( D \) is clique in \( G \). If \( V(H) \cap S \neq \emptyset \), then \( S_H = V(H) \setminus C \) and \( S_G = V(G) \text{ or } S_G = V(G) \setminus S_G \setminus V(G) \setminus \) is a dominating set of \( G \) where \( D \) is clique in \( G \). This shows statement (ii).

**Case 3.** Suppose that \( G \) and \( H \) are non-complete graphs. Since \( G \) and \( H \) are nontrivial connected graphs, there exists a nonempty subset \( C \) of \( V(G) \) and a nonempty subset \( D \) of \( V(H) \) such that \( C = V(G) \setminus S_G \) is a clique in \( G \) and \( D = V(H) \setminus S_H \) is a clique in \( H \). This shows statement (iii).

For the converse, consider first that statement (i) is satisfied.

**Case 1.** Suppose that \( S_G = \emptyset \).

**Subcase 1a.** If \( S_H = V(H) \), then \( S = S_H \) is a dominating set of \( G + H \). Since \( G \) is complete, \( V(G + H) \setminus S = V(G) \) is clique in \( G + H \).

**Subcase 1b.** If \( S_H = V(H) \setminus D \) is a dominating set of \( H \) where \( D \) is clique in \( H \), then \( S = S_H \) is a dominating set of \( G + H \). Since \( G \) is complete and \( D \) is clique in \( H \), \( V(G + H) \setminus S = V(G) \cup D \) is clique in \( G + H \).

**Case 2.** Suppose that \( S_G \neq \emptyset \). Let \( S_G = V(G) \setminus C \).

**Subcase 2a.** If \( S_H = V(H) \), then \( S = S_G \cup S_H \) is a dominating set of \( G + H \). Since \( G \) is complete, \( V(G + H) \setminus S = C \) is clique in \( G + H \).
Subcase 2b. If \( S_H = V(H) \setminus D \) is a dominating set of \( H \) where \( D \) is clique in \( H \), then \( S = S_G \cup S_H \) is a dominating set of \( G + H \). Since \( G \) is complete, \( C \) is clique in \( G \) and \( V(G + H) \setminus S = C \cup D \) is clique in \( G + H \). In either cases, \( S = S_G \cup S_H \) is an outer-clique dominating set of \( G + H \).

Next, consider that statement (ii) is satisfied. If \( H \) is a complete graph, then \( S_H = \emptyset \) or \( S_H = V(H) \setminus C \) for any nonempty set \( C \subseteq V(H) \), and \( S_G = V(G) \cup S_G = V(G) \setminus D \) is a dominating set of \( G \) where \( D \) is clique in \( G \). Using similar arguments in proving statement (i), \( S = S_G \cup S_H \) is an outer-clique dominating set of \( G + H \).

Finally, if statement (iii) is satisfied, then \( V(G) \setminus S_G \) is clique in \( G \) and \( V(H) \setminus S_H \) is clique in \( H \). Since \( S_G \subseteq V(G) \) and \( S_H \subseteq V(H) \), it follows that \( S = S_G \cup S_H \) is a dominating set of \( G + H \). Further, \( V(G + H) \setminus S = V(G + H) \setminus (S_G \cup S_H) = (V(G) \setminus S_G) \cup (V(H) \setminus S_H) \) is a clique in \( G + H \). Accordingly, \( S = S_H \cup S_H \) is an outer-clique dominating set of \( G + H \).

The next result is an immediate consequence of Theorem 2.8.

**Corollary 2.9.** Let \( G \) and \( H \) be nontrivial connected graphs, \( K = V(G) \setminus D \) where \( |D| = \max(|C| : C \text{ is clique in } G) \) is clique and \( K' = V(H) \setminus D' \) where \( |D'| = \max(|C'| : C' \text{ is clique in } H) \).

\[
\gamma_{cl}(G + H) = \begin{cases} 
2 & \text{if } G \text{ or } H \text{ is a noncomplete graph and } V(G) \setminus \{x\} \text{ is clique in } G \text{ and } V(H) \setminus \{y\} \text{ is clique in } H \\
|K'| & \text{if } H \text{ is a noncomplete graph and } G \text{ is a complete graph, } K' \text{ is a dominating set of } H
\end{cases}
\]

**Proof:** Suppose that \( G \) or \( H \) is a non-complete graph and \( V(G) \setminus \{x\} \) is clique in \( G \) and \( V(H) \setminus \{y\} \) is clique in \( H \). Consider first that \( G \) and \( H \) are both non-complete graphs. Let \( S = \{x, y\} \) where \( x \in V(G) \) and \( y \in V(H) \). Then \( S \) is a dominating set of \( G + H \) and \( V(G + H) \setminus S = V(G + H) \setminus \{x, y\} = (V(G) \setminus \{x\}) \cup (V(H) \setminus \{y\}) \) is clique in \( G + H \). Thus, \( S \) is an outer-clique dominating set of \( G + H \), that is, \( \gamma_{cl}(G + H) \leq |S| = 2 \). Since \( G \) and \( H \) are both non-complete graphs, \( G + H \) is a non-complete graph. In view of Theorem 2.5, \( \gamma_{cl}(G + H) \geq 2 \). Therefore, \( 2 \leq \gamma_{cl}(G + H) \leq 2 \), that is, \( \gamma_{cl}(G + H) = 2 \). The remaining cases (\( G \) complete and \( H \) non-complete or \( H \) complete and \( G \) non-complete) follow similarly.

Next, suppose that \( G \) is non-complete, \( H \) is complete, \( K \) is a dominating set of \( G \), and \( |D| = \max(|C| : C \text{ is clique in } G) \). Then \( K \) is a dominating set of \( G + H \). Since \( K = V(G) \setminus D \) where \( D \) is clique in \( G \) and \( H \) is complete, it follows that \( D \cup V(H) \) is clique in \( G + H \). Thus, \( V(G + H) \setminus K = (V(G) \setminus K) \cup (V(H) \setminus K) = D \cup V(H) \) is clique in \( G + H \). This implies that \( K \) is an outer-clique dominating set of \( G + H \), that is, \( \gamma_{cl}(G + H) \leq |K| \). Let \( K^o \) be a \( \gamma_{cl} \)-set in \( G + H \), that is, \( \gamma_{cl}(G + H) = |K^o| \). Then \( V(G + H) \setminus K^o \) is clique in \( G + H \). Let \( T = V(G + H) \setminus K^o \). Since \( |K^o| \leq |K| \) for all outer-clique \( K \) in \( G + H \), it follows that if \( |V(G + H)| = |K| = \max(|T| : T \text{ is clique in } G + H) \) then for all outer-clique \( K \) in \( G + H \).

\[
|V(G + H) \setminus K| = |V(G + H)| \setminus |K| 
\leq |V(G + H)| \setminus |K^o| 
= |V(G + H)| \setminus |K^o| \text{ for all outer-clique } K^o \text{ in } G + H 
= |T^o| \text{ for all clique } T^o \text{ in } G + H 
\leq \max(|T^o| : T^o \text{ is clique in } G + H) = |V(G + H) \setminus K|.
\]

Thus, \( |V(G + H) \setminus K| = |V(G + H) \setminus K^o| \), that is, \( |K| = |K^o| \). Therefore \( \gamma_{cl}(G + H) = |K^o| = |K| \).

Similarly, if \( H \) is non-complete, \( G \) is complete, \( K' \) is a dominating set of \( H \), and \( |D'| = \max(|C'| : C' \text{ is clique in } H) \). Then \( \gamma_{cl}(G + H) = |K'| \).

Since \( G + H \) is a complete graph and only if \( G \) is complete and \( H \) is complete, in view of Theorem 2.5 the following remark holds.

**Remark 2.10:** If \( G \) and \( H \) are complete graphs, then \( \gamma_{cl}(G + H) = 1 \).

**CONCLUSION**

An outer-clique dominating set is a new variant of domination in graphs. Hence this paper is a contribution to the development of domination theory in general. Since this is new, further investigations must be promoted to come up with coherent and substantial results of the parameter - an outer-clique domination number. Thus, the corona, lexicographic, and Cartesian products of two graphs of an outer-clique dominating sets are recommended for further study and scrutiny. Finally, domination in graphs is rich with immediate applications in the real world such as routing problems in internets, problems in electrical networks, data structures, neural and communication networks, protection and location strategies and many others. The outer-clique domination in graphs is not far from these applications.

**REFERENCES**

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