ON THE SOLUTION OF GENERALIZED FRACTIONAL KINETIC EQUATIONS

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Abstract: In view of the usefulness and a great importance of the kinetic equation in certain astrophysical problems the authors develop a new and further generalized form of the fractional kinetic equation involving Mittag-Leffler function and G-function. This new generalization can be used for the computation of the change of chemical composition in stars like the sun. The manifold generality of the Mittag-Leffler function and G-function is discussed in terms of the solution of the above fractional kinetic equation.

Saxena et al. [21, 22] derived the solutions of generalized fractional kinetic equations in terms of Mittaz-Leffler functions by the application of Laplace transform [9, 23]. The present work is extension of earlier work done by Saxena et al. [21, 22], and Chaurasia and Pandey [5].

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1. INTRODUCTION: The great importance of mathematical physics in distinguished astrophysical problems has attracted astronomers and physicists to pay more attention to available mathematical tools that can be widely used in solving several problems of physics and astrophysics. A spherically symmetric non-rotating, self-gravitating model of star like the Sun is assumed to be in thermal equilibrium and hydrostatic equilibrium. The star is characterized by its mass, luminosity, effective surface temperature, radius, central density and central temperature. The stellar structures and their mathematical models are investigated on the basis of above characters and some additional information related to the equation of nuclear energy generation rate and the opacity.

DEFINITION 1: SUMUDU TRANSFORM:

An integral transform, called the Sumudu transform was defined and studied by G.K. Watugala [25] to facilitate the process of solving differential and integral equations in the time domain, and for the use in various applications of system engineering and applied physics.

Despite the potential presented by this new operator, only few theoretical investigations have recently appeared in the literature which is still classical not referring the Sumudu transform in the space of generalized functions. Having scale and unit-preserving properties, the Sumudu transform may solve intricate problems in engineering mathematics and applied sciences without resorting to a new frequency domain. The Sumudu transform is also a theoretical dual of the Laplace transform with which interchange the images of the dirac and Heaviside functions.
In [2-4, 25], some fundamental properties of the Sumudu transform are established. It turns out that the Sumudu transform has very special and useful properties and it is useful in solving problems of science and engineering governing kinetic equations.

The Sumudu transform has been shown to be the theoretical dual of the Laplace transform. The Laplace transform is defined by

$$F \left( p \right) = \mathcal{L}_0 \left[ f \left( t \right) \right] = \int_0^\infty e^{-pt} f \left( t \right) \, dt, \quad \text{Re} \, p > 0. \quad (1.1)$$

The Sumudu transform is defined over the set of functions,

$$A = \left\{ f \left( t \right) \mid \exists M, \tau_1, \tau_2 > 0, \left| f \left( t \right) \right| < M e^{Ht} \text{ for } t \in -1 \times 0, \infty \right\} , \quad (1.2)$$

by

$$G \left( u \right) = S \left( f \left( t \right) \right) = \int_0^\infty e^{ut} f \left( t \right) \, dt, \quad u \in -\tau_1, \tau_2 : \quad (1.3)$$

where \( M \) is a real finite number and \( \tau_1 \) and \( \tau_2 \) can be finite or infinite [25].

Hence, \( G \left( u \right) \) is called as the Sumudu transform of \( f \left( t \right) \). It is obvious that this is a linear operator. It can be easily verified that in (1.3) the function \( G \left( u \right) \) keeps the same units as \( f \left( t \right) \), for any real or complex number \( \lambda \) it gives that

$$S \left[ f \left( \lambda t \right) \right] = G \left( \lambda u \right).$$

The Sumudu and Laplace transforms exhibit a duality relation that may be expressed either as

(i) \[ G \left( \frac{1}{u} \right) = u F \left( u \right) \quad \text{or} \quad G \left( u \right) = \frac{1}{u} F \left( \frac{1}{u} \right), \quad (1.4) \]

(ii) \[ F \left( \frac{1}{p} \right) = p G \left( p \right) \quad \text{or} \quad F \left( p \right) = \frac{1}{p} G \left( \frac{1}{p} \right). \quad (1.5) \]

The Sumudu transform is connected to the \( p \)-multiplied Laplace transform (see [16]).

This however is no way diminishes its importance or usefulness. In fact, we show that the Sumudu transform has deeper connections with the Laplace transform than previously established.

The discrete analog of the Sumudu integral transform (1.3) is defined for power series functions \( f \left( t \right) = \sum_{k=0}^\infty a_k t^k \), having an interval of convergence containing \( t = 0 \) as follows

$$G \left( u \right) = \sum_{k=0}^\infty k! a_k u^k \quad \text{for} \quad u \in -\tau_1, \tau_2 . \quad (1.6)$$

So, the linear function \( f \left( t \right) = a_0 + a_1 t \) transforms to itself, \( G \left( u \right) = a_0 + a_1 u = f \left( u \right) \). However, the power series

$$f \left( t \right) = \sum_{k=0}^\infty -1 \frac{a t^k}{k!} = e^{-at} , \quad (1.7)$$

transforms to the geometric series

$$G \left( u \right) = \sum_{k=0}^\infty -1 \frac{a u^k}{k!} = \frac{1}{1+au} \quad \text{with} \quad u \in \left[ -\frac{1}{a} , \frac{1}{a} \right] .$$

**Inverse Sumudu Transform:** Up to null functions, the inverse discrete Sumudu transform, \( f \left( t \right) \), of the power series

$$G \left( u \right) = \sum_{n=0}^\infty b_n u^n ,$$

is given by

$$f \left( t \right) = \sum_{n=0}^\infty b_n t^n.$$
2. GENERALIZED MITTAG-LEFFLER FUNCTION

In 1903, the Swedish mathematician Gosta Mittag-Leffler [17, 18] introduced the function \( E_\alpha z \), defined by

\[
E_\alpha z = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha \in \mathbb{C}, \text{Re} \, \alpha > 0.
\]  

(2.1)

The Mittag-Leffler function \( E_\alpha z \) was studied by Wiman [26] who defined the function

\[
E_{\alpha, \beta} z = \sum_{n=0}^{\infty} \frac{z^n}{\alpha n + \beta}, \quad \alpha, \beta \in \mathbb{C}, \text{Re} \, \alpha > 0, \text{Re} \, \beta > 0.
\]  

(2.2)

The function \( E_{\alpha, \beta} z \) is now known as Wiman function, which was later studied by Agarwal [1] and others.

The generalization of (2.2) was introduced by Prabhakar [19] in terms of the series representation

\[
E_{\alpha, \beta}^\gamma z = \sum_{n=0}^{\infty} \frac{\gamma_n z^n}{\Gamma(\alpha n + \beta)} n!, \quad \alpha, \beta, \gamma \in \mathbb{C}, \text{Re} \, \alpha > 0, \text{Re} \, \beta > 0.
\]  

(2.3)

where \( \gamma_n \) is Pochhammer’s symbol, defined by

\[
\gamma_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} = 1, \text{ for } n=0, n \neq 0, n \in \mathbb{N}, n \in \mathbb{C},
\]  

(2.4)

It is an entire function of order [19] \( \rho = \left[ \text{Re} \, \alpha \right]^{-1} \). Some special cases of (2.3) are

\[
E_\alpha z = E_{\alpha, 1}^1 z, \quad E_{\alpha, \beta} z = E_{\alpha, \beta}^1 z, \quad \Phi \beta, \gamma; z = \Gamma \gamma E_{\alpha, \beta}^\gamma z.
\]  

(2.5)

where \( \Phi \beta, \gamma; z \) is Kummer’s confluent hypergeometric function [12]. Mellin-Barnes integral representation for the function (2.3) is given by [22]:

\[
E_{\alpha, \beta}^\gamma z = \frac{1}{2\pi i} \int_{\gamma} \Gamma -s \Gamma \gamma + s \frac{-z^s}{\Gamma \beta + s\alpha} \, ds,
\]  

(2.6)

where \( i = \sqrt{-1} \).

Remark 1:- A detailed account of Mittag-Leffler functions and their Applications can be found in the monograph by Haubold, Mathai and Saxena [12].

The following integral gives the Sumudu transform of \( E_{\alpha, \beta}^\gamma z \):

\[
\int_{0}^{\infty} e^{-ut} t^{\beta-1} \Gamma \gamma E_{\alpha, \beta}^\gamma w u t^{\alpha} \, dt = u^{\beta-1} \left( 1 - w u^{\alpha - \gamma} \right).
\]  

(2.7)
where $\Re u > |w|^{1/\Re \alpha}$, $\Re \beta > 0$, $\Re u > 0$, $u \in (-\tau_1, \tau_2)$, $\int f(t) \leq M e^{V(t)}$, which can be established by means of the Gamma function

$$\int_0^\alpha e^{-t} t^{\nu-1} = \Gamma(\nu), \quad \Re \nu > 0,$$

(2.8)

and the binomial formula

$$1 - z^{-a} = \sum_{n=0}^\infty \frac{a^n z^n}{n!}.$$

(2.9)

The Laplace transform of $E_{\alpha,\beta}^{\gamma} t^\alpha$ is as follows

$$\int_0^\alpha e^{-pt} t^{\beta-1} E_{\alpha,\beta}^{\gamma} t^\alpha \ dt = p^{-\beta} 1 - ap^{-\alpha}^{-\gamma}.$$

(2.10)

If we set $\gamma = 1$, (2.7) reduces to

$$\int_0^\alpha e^{-ut} u^{\beta-1} E_{\alpha,\beta} t^\alpha \ dt = u^{\beta-1} 1 - wu^{-\alpha}^{-1}.$$

(2.11)

where $|u| > |w|^{1/\Re \alpha}$, $\Re \beta > 0$, $\Re u > 0$, $u \in (-\tau_1, \tau_2)$, $\int f(t) \leq M e^{V(t)}$.

Now, we recall the definition of Riemann-Liouville integral operator [24] of order $\alpha \in C$.

$$a D_x^\alpha f \ x = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \ dt, \quad x > a, \Re \alpha > 0,$$

(2.12)

in the form

$$0 D_t^\nu f \ t = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s) \ ds, \quad t > 0, \Re \nu > 0,$$

(2.13)

with $a D_x^\alpha f \ t = f(t)$.

Fractional derivative for $\alpha > 0$ is defined as

$$0 D_t^\alpha f \ t = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(u) \ du}{(t-u)^{\alpha+n}} \ n = \Re \alpha + 1;$$

where $\Re \alpha$ stand for the integral part of $\Re \alpha$.

If we apply convolution theorem for Sumudu transform [2-4], we observe that (2.13) can be written in the following form:

$$S_0 D_t^\nu f \ t = S_0 \left\{ \frac{t^{\nu-1}}{\Gamma(\nu)} \right\} \cdot f(t) \ t = u^\nu G u.$$

(2.14)

In the following, we present the solution of two generalized fractional kinetic equations. The results are obtained in terms of generalized Mittag-Leffler functions in a compact form and can be worked out easily. A detailed account of the fractional integral operators and their applications is available in Ref. [24].

3. SOLUTION OF FRACTIONAL KINETIC EQUATIONS
Theorem 1. If $c > 0$, $\nu > 0$, $\mu > 0$, $\gamma > 0$, then for the solution of the equation
\[
N_t = N_0 t^{\alpha-1} E_{\nu,\beta}^{\gamma, \alpha} - c^\nu D_t^{-\nu} N_t ,
\]  
there holds the formula
\[
N_t = N_0 t^{\alpha-2} E_{\nu,\beta}^{\gamma, \alpha} - c^\nu D_t^{-\nu} ,
\]  
where $E_{\nu,\beta}^{\gamma, \alpha}$ is the generalized Mittag-Leffler function.

Proof.

Applying the Sumudu transform to the both sides of (3.1) and using (2.14), we get
\[
S[N_t] = N_0 S \left[ t^{\alpha-1} E_{\nu,\beta}^{\gamma, \alpha} - c^\nu D_t^{-\nu} N_t \right]
\]
\[
N^* u = N_0 \left[ \frac{u^{\alpha-1}}{1 + \nu c^\nu} \right] - c^\nu N^* u
\]
\[
N^* u = N_0 \left[ \frac{u^{\alpha-1}}{1 + \nu c^\nu} \right]^{\gamma+1} .
\]  
Using the relation $S^{-1} u^\nu = \frac{t^{\nu-1}}{\Gamma \nu}$, Re $\nu > 0$, Re $u > 0$, and taking the Sumudu inverse of (3.3), we have,
\[
S^{-1} N^* u = N_0 S^{-1} \left[ \frac{u^{\alpha-1}}{1 + \nu c^\nu} \right] = N_0 \left[ \sum_{n=0}^{\infty} \frac{-1^n \gamma + 1 \nu c^\nu}{n!} \right]
\]  
\[
= N_0 \sum_{n=0}^{\infty} \frac{-1^n \gamma + 1 \nu c^\nu}{n!} S^{-1} u^{\alpha+n-1}
\]
\[
N_t = N_0 \sum_{n=0}^{\infty} \frac{-1^n \gamma + 1 \nu c^\nu}{n!} \frac{t^{\nu+n-2}}{\Gamma \mu + vn - 1} = N_0 t^{\alpha-2} E_{\nu,\beta}^{\gamma, \alpha} - c^\nu D_t^{-\nu}. \]
This completes the proof of Theorem 1.

Now, if we follow the definition of $E_{\alpha,\beta}^{\gamma, \alpha}$ given by (2.3) and set $\gamma = 1$, then, we arrive at the following result:

Corollary 1.1. If $c > 0$, $\nu > 0$, $\mu > 0$ then for the solution of
\[
N_t = N_0 t^{\alpha-1} E_{\nu,\beta}^{\gamma, \alpha} - c^\nu D_t^{-\nu} N_t ,
\]  
there holds the relation
\[
N^* u = N_0 \left[ \frac{u^{\alpha-1}}{\nu} \left( E_{\nu,\beta}^{\gamma, \alpha} - c^\nu D_t^{-\nu} \right) \right] .
\]  
\[
(3.5)
\]
Proof.

Applying the Sumudu transform to the both sides of (3.4) and using (2.14), we get

\[
S\left[ N \ t \right] = N_0 S\left[ t^{\mu-1} E_{\nu,\mu} - c^\nu t^\nu \right] - c^\nu S\left[ _0D_t^{-\nu} N \ t \right]
\]

\[
N^* \ u = N_0 \left[ \frac{u^{\nu-1}}{1 + uc^\nu} \right] - c^\nu N^* \ u \ , \text{ or } N^* \ u = N_0 \left[ \frac{u^{\nu-1}}{1 + uc^\nu} \right]^2 . \tag{3.6}
\]

Using the relation \( S^{-1} u^\nu = \frac{t^{\nu-1}}{\Gamma \nu} \), \( \text{Re} \ \nu > 0 \), \( \text{Re} \ u > 0 \), and taking the Sumudu inverse of (3.6), we have,

\[
S^{-1} N^* \ u = N \ t = N_0 \sum_{n=0}^{\infty} \frac{-1}{n!} \frac{n+1}{\mu+vn-1} S^{-1} u^{\nu n-1}
\]

\[
= N_0 \sum_{n=0}^{\infty} \frac{-1}{n+1} \frac{n+1}{\mu+vn-1} E_{\nu,\mu}^{-\nu} = N_0 t^{\mu-2} \sum_{n=0}^{\infty} \frac{-1}{n+1} \frac{ct^\nu}{\mu+vn-1}
\]

\[
= N_0 t^{\mu-2} \sum_{n=0}^{\infty} \frac{-1}{\nu} \frac{1}{\mu+vn-2} + 2 + v - \mu \] \frac{ct^\nu}{\mu+vn-1}
\]

\[
= N_0 t^{\mu-2} \left[ \frac{E_{\nu,\mu-2} c^\nu t^\nu + 2 + v - \mu \ E_{\nu,\mu-1} c^\nu t^\nu}{v} \right] .
\]

This completes the proof of (3.5).

If we set \( \nu = 2 \) in Theorem 1, then we obtain the following:

**Corollary 1.2.** If \( c > 0 \), \( v > 0 \), \( \mu > 0 \), then for the solution of

\[
N \ t = N_0 t^{\mu-1} E_{\nu,\mu}^2 - c^\nu t^\nu = -c^\nu _0D_t^{-\nu} N \ t , \tag{3.7}
\]

there holds the relation

\[
N \ t = \frac{N_0}{2v^2} \left[ E_{\nu,\mu-3} c^\nu t^\nu + 3v - 2\mu + 5 \ E_{\nu,\mu-2} c^\nu t^\nu + 2 + v^2 + \mu^2 - 2\mu + 1 + 6v - 2\mu - 3v\mu + 3 \ E_{\nu,\mu-1} c^\nu t^\nu \right] . \tag{3.8}
\]

Proof.

Applying the Sumudu transform to the both sides of (3.7) and using (2.14), we get

\[
S\left[ N \ t \right] = N_0 S\left[ t^{\mu-1} E_{\nu,\mu}^2 - c^\nu t^\nu \right] - c^\nu S\left[ _0D_t^{-\nu} N \ t \right]
\]
\[ N^* u = N_0 \left[ \frac{u^{\mu-1}}{1 + \frac{uc}{v}} \right] c^\nu u^\nu N^* u, \quad \text{or} \quad N^* u = N_0 \left[ \frac{u^{\mu-1}}{1 + \frac{uc}{v}} \right]^3. \]  \tag{3.9}

Using the relation \( S^{-1} u^\nu = \frac{t^{\nu-1}}{\Gamma \nu}, \quad \text{Re} \nu > 0, \text{Re} u > 0, \) and taking the Sumudu inverse of (3.9), we have,

\[ S^{-1} N^* u = N t = N_0 \sum_{n=0}^\infty -1^n \frac{3^n c^{vn}}{n!} S^{-1} u^{vm-1} \]

\[ = N_0 \sum_{n=0}^\infty -1^n \frac{n+1}{2} \frac{n+2}{n!} \frac{t^{\nu+vm-2}}{\Gamma \mu + vn - 1} \]

\[ = \frac{N_0 t^{\mu-2}}{2v^2} \left[ E_{\nu,\mu-3} - c^\nu t^\nu + 3v - 2\mu + 5 E_{\nu,\mu-2} - c^\nu t^\nu \right. \]

\[ + 2v^2 + \mu^2 - 2\mu + 1 + 6v - 2\mu - 3v\mu + 3 E_{\nu,\mu-1} - c^\nu t^\nu \].

This completes the proof of (3.8).

**Theorem 2.** If \( \nu > 0, c > 0, d > 0, \mu > 0, \text{Re} u > |d|^{1/\nu}, c \neq d \), then for the solution of the equation

\[ N t = N_0 t^{\mu-1} E_{\nu,\mu} - d' t^\nu = -c^\nu D_t^{-\nu} N t, \] \tag{3.10}

there holds the formula

\[ N t = \frac{N_0}{c^\nu - d^\nu} t^{\mu-\nu-2} \left[ E_{\nu,\mu-\nu-1} - d' t^\nu - E_{\nu,\mu-\nu-1} - c^\nu t^\nu \right]. \] \tag{3.11}

where \( E_{\nu,\mu} \) is the generalized Mittag-Leffler function (also known as Agarwal function) [1].

**Proof.**

Applying the Sumudu transform to the both sides of (3.10) and using (2.14), we get

\[ S \begin{bmatrix} N t \end{bmatrix} = N_0 S \left[ t^{\mu-1} E_{\nu,\mu} - d' t^\nu \right] - c^\nu S \left[ _0D_t^{-\nu} N t \right] \]

\[ N^* u = N_0 \left[ \frac{u^{\mu-1}}{1 + \frac{du}{v} + cu^\nu} \right] \]

\[ = \frac{N_0}{c^\nu - d^\nu} u^{\mu-\nu-1} \left[ \sum_{n=0}^\infty -1^n d^u^{vn} - \sum_{n=0}^\infty -1^n cu^{vn} \right]. \] \tag{3.12}

Using the relation \( S^{-1} u^\nu = \frac{t^{\nu-1}}{\Gamma \nu}, \quad \text{Re} \nu > 0, \text{Re} u > 0, \) and taking the Sumudu inverse of (3.12), we have,
\[ S^{-1} N^* u = \frac{N_0}{c^v - d^v} \left[ \sum_{n=0}^{\infty} -1 \frac{1}{n!} c^v s^{-1} u^{\mu - v + n - 1} \right] \]
\[ = \frac{N_0}{c^v - d^v} \left[ \sum_{n=0}^{\infty} -1 \frac{1}{\Gamma \mu + vn - v - 1} \right] \]
\[ N^* t = \frac{N_0}{c^v - d^v} \left[ E^v \mu - v - 1 - d^v t^v - E^v \mu - v - 1 - c^v t^v \right]. \]

This completes the proof of Theorem 2.

When \( \mu = v + 2 \), Theorem 2 reduces to

Corollary 2.1. If \( v > 0, c > 0, d > 0, \Gamma > 0, \Re u > |d|^{1/\alpha}, c \neq d \), then for the solution of

\[ N^* t = \frac{N_0}{c^v - d^v} \left[ E^v \mu - v - 1 - d^v t^v - E^v \mu - v - 1 - c^v t^v \right]. \]

the following result holds

On the other hand if \( d \to 0 \) in Theorem 2, we arrive at the following result:

Corollary 2.2. If \( v > 0, c > 0, \mu > 0, \Re u > |d|^{1/\alpha} \), then for the solution of

\[ N^* t = \frac{N_0}{\Gamma \mu} \left[ \frac{1}{\Gamma \mu - v - 1} - E^v \mu - v - 1 - c^v t^v \right]. \]

the following result holds

REFERENCES:


