ON PAIRS OF DISJOINT DOMINATING SETS IN THE COMPOSITION OF GRAPHS

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Abstract: In this paper, we investigate pairs of disjoint dominating sets $A$ and $B$ in the composition of graph, where $B$ is either an independent or a total dominating set in the composition of graph.

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1 INTRODUCTION

Throughout this study, we only consider graphs which are finite, simple and undirected. The symbols $V(G)$ denotes the vertex set and $E(G)$ denotes the edge set of $G$. The order of $G$ refers to the cardinality of $V(G)$ and the size of $G$ refers to the cardinality of $E(G)$. The symbol $|V(G)|$ denotes the order of $G$ and $|E(G)|$ denotes the size of $G$. If $|E(G)| = 0$, then $G$ is an empty graph. An empty graph of order $n$ is denoted by $K_n$. If $V(G)$ is singleton, $G$ is called a trivial graph.

The composition $G[H]$ of $G$ and $H$ is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ if and only either $uu' \in E(G)$ or $v = u'$ and $vv' \in E(H)$. For any $v \in V(G)$, $G - v$ is the resulting graph after removing from $G$ the vertex $v$ and all edges of $G$ incident to $v$.

Two distinct vertices $u$ and $v$ of $G$ are neighbors in $G$ if $uv \in E(G)$. The closed neighborhood $N_c[v]$ of a vertex $v$ of $G$ is the set consisting of $v$ and every neighbor of $v$ in $G$. A dominating set in $G$ is any $S \subseteq V(G)$ for which $N_c[S] = V(G)$. The minimum cardinality of a dominating set is called the domination number of $G$, denoted by $\gamma(G)$. Any dominating set in $G$ of cardinality $\gamma(G)$ is referred to as a $\gamma$-set in $G$. A dominating set $S$ in $G$ is an independent dominating set if $uv \notin E(G)$ for all $u, v \in S$. The minimum cardinality of an independent dominating set in $G$ is called the independence domination number of $G$, denoted by $\gamma'_I(G)$. Any independent dominating set in $G$ of cardinality $\gamma'_I(G)$ is referred to as a $\gamma_I$-set in $G$. A total dominating set $S$ if for each $v \in S$ there is $v' \in S$ such that $uv \in E(G)$. The minimum cardinality of a total dominating set is called the total domination number of $G$, denoted by $\gamma_t(G)$. Any total dominating set in $G$ of cardinality $\gamma_t(G)$ is referred to as a $\gamma_t$-set in $G$. The symbols $D(G), \mathcal{J}(G)$ and $T(G)$ are used to denote the collection of all dominating sets, the collection of all independent dominating sets, and the collection of all total dominating sets in $G$, respectively. The reader may refer to [1, 3, 5, 6, 7, 8, 10, 11, 24, 25] for the fundamental concepts of domination theory, and to [3, 12, 25] for its applications.

Domination is one of the most well-studied concepts in graph theory (see [11]). The reader is referred to [1, 3, 5, 6, 7, 8, 10, 11, 24, 25] for the fundamental concepts and recent developments of the domination theory, and to [3, 12, 15, 25] for its various applications.

In 1962, O. Ore gave the classical result which can be stated as follows: If a graph $G$ has no isolated vertices and $S$ is a minimum dominating set, then $V(G) \setminus S$ is a dominating set in $G$. It has motivated the introduction of the concept of inverse domination (by V.R. Kulli and S.C. Sigarkanti [21]) as well as the concept of disjoint domination (by S.M. Hedetniemi, S.T. Hedetniemi, R.C. Laskar, L. Markus, and P.J. Slater [14]). A subset $S \subseteq V(G)$ is an inversedominating set in $G$ if $S$ is a dominating set in $G$ and there is a minimum dominating set $D$ in $G$ such that $S \cap D = \emptyset$. The minimum cardinality of an inverse dominating set in $G$ is the inverse domination number of $G$, which is denoted by $\gamma^*(G)$. A pair $(S, D)$ of dominating sets in $G$ is a dd-pair if $S \cap D = \emptyset$. We denote by $\mathcal{D}_D(G)$ the collection of all dd-pairs in $G$. The minimum sum $|S| + |D|$ among all dd-pairs $(S, D)$ in $G$ is the disjoint domination number of $G$, which is denoted by $\gamma_D(G)$. That is,
\[ \gamma\gamma(G) = \min\{|S| + |D|: (S, D) \in \mathcal{D}(G)\}. \]

A dd-pair \((S, D)\) with \(|S| + |D| = \gamma\gamma(G)\) is called a \(\gamma\gamma\)-pair.

Inverse domination is studied further in [9, 19, 22]. Disjoint domination is also further investigated in [13, 15, 16, 20, 23].

2 COMPOSITION OF GRAPHS

For any connected graphs \(G\) and \(H\), if \(S \subseteq V(G[H])\) is a \(\gamma\)-set in \(G[H]\), then \(S\) is a \(\gamma\)-set in \(G\). Consequently, \(\gamma(G) \leq \gamma(G[H])\).

**Theorem 2.1** [6] Let \(G\) and \(H\) be connected graphs. Then \(C = U_{x \in S} \{x\} \times T_x \subseteq V(G[H])\), where \(S \subseteq V(G)\) and \(T_x \subseteq V(H)\) for every \(x \in S\), is an dominating set in \(G[H]\) if and only if either

(i) \(S\) is a total dominating set in \(G\) or

(ii) \(S\) is a dominating set in \(G\) and \(T_x\) is a dominating set in \(H\) for every \(x \in S\).

**Corollary 2.3** Let \(G\) and \(H\) be connected graphs. A subset \(C = U_{x \in S} \{x\} \times T_x \subseteq V(G[H])\), is a \(\gamma_i\)-set in \(G[H]\) if and only if \(S\) is a \(\gamma_i\)-set in \(G\) and \(T_x\) is a dominating set in \(H\) for every \(x \in S\).

**Proof :** Suppose that \(C = U_{x \in S} \{x\} \times T_x\) is a \(\gamma_i\)-set in \(G[H]\). By Theorem 2.2, \(S\) is an independent dominating set in \(G\) and \(T_x\) is an independent dominating set in \(H\) for each \(x \in S\). Suppose that \(S^*\) is a \(\gamma_i\)-set in \(G\), and let \(D \subseteq V(H)\) be a \(\gamma_i\)-set in \(H\). Define \(C^* = U_{x \in S^*} \{x\} \times D\). By Theorem 2.2, \(C^*\) is an independent dominating set in \(G[H]\). Since \(C^*\) is a \(\gamma_i\)-set in \(G[H]\), \(|C| \leq |C^*|\). On the other hand, \(|C| = |S^*||D| \leq |S||D| \leq |C|\). Thus, \(|C| = |C^*|\), and consequently, \(|S| = |S^*|\) and \(|D| = |T_x|\) for all \(x \in S\). This means that \(S\) is a \(\gamma_i\)-set in \(G\) and \(T_x\) is a \(\gamma_i\)-set in \(H\) for every \(x \in S\).

Similar arguments will prove the converse.

**Lemma 2.4** Let \(G\) and \(H\) be connected nontrivial graphs such that \(V(H)\) is dominated in \(H\) by a vertex \(v \in V(H)\). If \(A \subseteq V(G)\) is an inverse independent dominating set in \(G\), then \(A \times \{v\}\) is an inverse independent dominating set in \(G[H]\).

**Proof :** Let \(B \subseteq V(G)\) be a \(\gamma_i\)-set in \(G\) and \(A \subseteq V(G) \setminus B\) a dominating set in \(G\). By Corollary 2.3, \(B \times \{v\}\) is a \(\gamma_i\)-set in \(G[H]\). Also, by Theorem 2.1, \(A \times \{v\}\) is a dominating set in \(G[H]\). Since \((A \times \{v\}) \cap (B \times \{v\}) = \emptyset\), \(A \times \{v\}\) is an independent inverse dominating set in \(G[H]\).

**Corollary 2.5** Let \(G\) and \(H\) be connected nontrivial graphs with \(\gamma(H) = 1\). Then

\[ \gamma(G) \leq \gamma_i(G[H]) \leq \gamma_i(G). \]  

Define \(S^* = S \setminus N_G(S)\) for any \(S \subseteq V(G)\).

**Theorem 2.6** Let \(G\) and \(H\) be nontrivial connected graphs with \(\gamma(H) = 1\).

(i) If \(H\) has (at least) two distinct vertices each of which dominates \(V(H)\), then \(\gamma_i(G[H]) = \gamma(G)\).

(ii) If \(H\) has a unique vertex that dominates \(V(H)\), then...
\[ y_i'(G[H]) = \min(|A| + |A' \cap B| (y'(H) - 1) : A \in D(G), B \in J(G) \]

with \( y_i(G) = |B| \).

**Proof:** (i) Suppose that \( H \) has two distinct vertices \( u \) and \( v \) such that \( N[u] = V(H) = N[v] \), and let \( A, B \subseteq V(G) \) be a \( y_i \)-set and a \( y_i \)-set, respectively, in \( G \). Then \( S = A \times \{u\} \cup B \times \{v\} \) is a \( y_i \)-set and a \( y_i \)-set, respectively, in \( G[H] \). Since \( S \cap D = \emptyset \), \( S \) is a \( y_i \)-set in \( G[H] \). Hence, \( y_i'(G[H]) \leq |S| = |A| = y(G) \). The desired equality follows from Inequality 1.

Suppose that \( H \) has a unique vertex \( v \) that dominates \( V(H) \), and let

\[ \alpha = \min(|A| + |A' \cap B| (y'(H) - 1) : A \in D(G), B \in J(G) with y_i(G) = |B|). \]

Let \( A \) and \( B \) be a dominating set and \( y_i \)-set, respectively, in \( G \), and let \( v \in V(H) \) such that \( N[v] = V(H) \). Choose \( w \in V(H) \setminus \{v\} \) and a \( y \)-set \( C \subseteq V(H) \) in \( H \). Since \( H \) has a unique dominating set, namely \( \{v\}, v \notin C \).

Define \( D = B \times \{v\} \) and

\[ S = (\cup_{u \in A \setminus B} \{(u,v)\}) \cup (\cup_{u \in (A \setminus A') \cap B} \{(u,v)\}) \cup (\cup_{u \in A' \cap B} \{(u,x \times C)\}). \]

By Corollary 2.3, \( D \) is a \( y_i \)-set in \( G[H] \). Let \( u \in A' \). Then \( T_u = \{x \in V(H) : (u,v) \in S\} \) is either \( C \) or \( \{v\} \). In any case, \( T_u \) is a dominating set in \( H \). By Theorem 2.1, \( S \) is a dominating set in \( G[H] \). Since \( S \cap D = \emptyset \), \( S \) is an inverseindependent dominating set in \( G[H] \). Thus,

\[ y_i'(G[H]) \leq |S| = |A| + |A' \cap B| (y'(H) - 1). \]

Since \( A \) and \( B \) are arbitrary, \( y_i'(G[H]) \leq \alpha \).

Let \( (S,D) \) be a \( y_i \)-pair in \( G[H] \) such that \(|D| = y_i(G[H]) \) and \(|S| = y_i'(G[H]) \). By Theorem 2.1, \( S = \cup_{u \in A} \{(u) \times T_u\} \) and \( D = \cup_{u \in B} \{(u) \times T_u\} \) for some dominating sets \( A \) and \( B \) in \( G \). More particularly, by Corollary 2.3, \( B \) is a \( y_i \)-set in \( G \). Since \( y(H) = 1 \), Corollary 2.7 implies that \( |T_u| = 1 \) for all \( u \in B \) and \(|D| = |B| = y_i(G) \). Since \( S \) is a \( y_i \)-set, \( |T_u| = 1 \) for all \( u \in A \setminus B \), in which case, we may assume that \( T_u = \{v\} \subseteq V(H) \) where \( N[v] = V(H) \). Since \( S \cap D = \emptyset \), for all \( u \in A' \cap B \), if \( (u,w) \in D \), then \( (u,w) \notin S \). Moreover, in view of Theorem 2.1(ii), for each such \( u \), \( T_u = \{x \in V(H) : (u,x) \in S\} \) is a \( y_i \)-set in \( H \). Thus,

\[ |S| = |\cup_{u \in A \setminus B} \{(u) \times T_u\}| + |\cup_{u \in (A \setminus A') \cap B} \{(u) \times T_u\}| + |\cup_{u \in A' \cap B} \{(u) \times T_u\}| \]

\[ \geq |A \setminus (A' \cap B)| + |A' \cap B| (y'(H)) \]

\[ = |A| + |A' \cap B| (y'(H) - 1) \]

So that \( y_i'(G[H]) \geq \alpha \).

**Corollary 2.7** Let \( G \) and \( H \) be connected nontrivial graphs. Suppose that \( H \) has a unique vertex that dominates \( V(H) \). Then

(i) \( y_i'(G[H]) = y_i'(G) \) if and only if \( G \) has a \( y_i \)-set \( A_0 \) such that

\[ |A_0| \leq |A \cap B| (y'(H) - 1) + |A| \]

for all dominating sets \( A \) and \( y_i \)-sets \( B \) in \( G \).

(ii) If \( y_i'(G) = y(G) \), then \( y_i'(G[H]) = y(G) \).

**Proof:** (i) Let \( A_0 \subseteq V(G) \) be a \( y_i \)-set in \( G \) such that \(|A_0| \leq |A \cap B| (y'(H) - 1) + |A| \) for all dominating sets \( A \) and \( y_i \)-sets \( B \) in \( G \). By Theorem 2.6 and Inequality 1 in Corollary 2.5, \( y_i'(G[H]) = |A_0| \leq y_i'(G[H]) \leq y_i'(G) \). The converse is clear.

(ii) Let \( B \subseteq V(G) \) be a \( y_i \)-set in \( G \). Put \( A = B \). Since \( A' = \emptyset \) and \( y(G) = |A| = |A| + |A' \cap B| (y'(H) - 1), \) Theorem 2.6 implies \( y_i'(G[H]) \leq |A| = y(G) \). The desired equality follows from Inequality 1 in Corollary 2.5.
Theorem 2.8 Let $G$ and $H$ be connected nontrivial graphs with $\gamma(H) = 1$. Then

$$2\gamma(G) \leq \gamma\gamma_t(G[H]) \leq \gamma\gamma_t(G). \ (2)$$

More precisely,

(i) if $H$ has (at least) two distinct vertices each of which dominates $V(H)$, then $\gamma\gamma_t(G[H]) = 2\gamma(G)$; and

(ii) if $H$ has a unique vertex that dominates $V(H)$, then

$$\gamma\gamma_t(G[H]) = \min\{|A| + |B| + |A^° \cap B|(\gamma'(H) - 1): A \in \mathcal{D}(G), B \in \mathcal{I}(G)\}.$$

Proof: There exists $v \in V(H)$ such that $N_H[v] = V(H)$. Let $(A, B)$ be a $\gamma\gamma_t$-pair in $G$. Then $(A \times \{v\}, B \times \{v\})$ is a $d_i$-pair in $G[H]$. Thus, $\gamma\gamma_t(G[H]) \leq |A \times \{v\}| + |B \times \{v\}| = |A| + |B| = \gamma\gamma_t(G)$.

If $H$ has two distinct vertices that both dominate $V(H)$, then Theorem 2.6(i) implies

$$2\gamma(G) \leq \gamma\gamma_t(G[H]) \leq \gamma(G[H]) + \gamma'(G[H]) = 2\gamma(G).$$

Suppose that $H$ has a unique vertex $v$ that dominates $V(H)$. Let

$$\alpha = \min\{|A| + |B| + |A^° \cap B|(\gamma'(H) - 1): A \in \mathcal{D}(G), B \in \mathcal{I}(G)\}.$$

Let $w \in V(H) \setminus \{v\}$, $(X, Y)$ a $d_i$-pair in $H$, and let $A$ and $B$ be a dominating set and an independent dominating set, respectively, in $G$. Define

$$S = (U_{u \in A \setminus B} \{(u, v)\}) \cup (U_{u \in (A^\circ \cap B)} \{(u, w)\}) \cup (U_{u \in A^\circ \setminus B} \{(u) \times X\}),$$

and $D = (U_{u \in A^\circ \setminus B} \{(u) \times Y\}) \cup (U_{u \in B \setminus (A^\circ \cap B)} \{(u, v)\})$. By Theorem 2.1, $S$ is a dominating set in $G[H]$. By Theorem 2.2, $D$ is an independent dominating set in $G$. Moreover, $S \cap D = \emptyset$. Thus,

$$\gamma\gamma_t(G[H]) \leq |S| + |D| = |A| + |B| + |A^\circ \cap B|(|X| + |Y| - 2).$$

Since $X$ and $Y$ are arbitrary $\gamma\gamma_t(G[H]) \leq |A| + |B| + |A^\circ \cap B|\gamma'(H) - 2).$

Write $H = K_1 + H^*$, where $\gamma(H^*) \geq 2$. So, $(\gamma\gamma_t(H) - 2) = \gamma'(H) - 1$.

Thus,

$$\gamma\gamma_t(G[H]) \leq |A| + |B| + |A^\circ \cap B|\gamma'(H) - 1)$$

Since $A$ and $B$ are arbitrary, $\gamma\gamma_t(G[H]) \leq \alpha$.

To prove the converse, let $(S, D)$ be a $\gamma\gamma_t$-pair in $G[H]$. There exists an adominating set $A$ in $G$ and an independent dominating set $B$ in $G$ such that $S = U_{u \in A} \{(u) \times T_u\}$ and $D = U_{v \in B} \{(u) \times T_v\}$. If $A$ is not a total dominating set in $G$, then $T_u$ is a dominating set in $H$. Also, $T_u$ is an independent dominating set in $H$ for all $u \in B$. In particular, for each $u \in A \cap B, X = \{y \in V(H): (u, y) \in S\}$ and $Y = \{y \in V(H): (u, y) \in D\}$ constitute a $d_i$-pair in $H$. Since $(S, D)$ is a $\gamma\gamma_t$-pair in $G[H]$, we have for each $u \in A \cap B, |X| + |Y| \geq \gamma\gamma_t(H)$. For each $u \in (A \setminus A^\circ) \cap B, \{y \in V(H): (u, y) \in D\} = \{v\}$, and for each $u \in B \setminus A, \{y \in V(H): (u, y) \in D\} = \{v\}$. Thus,

$$\gamma\gamma_t(G[H]) = |S| + |D| \geq |A| + |B| + |A^\circ \cap B|\gamma'(H) - 2) \geq \alpha.$$

This proves Statement (ii). ■

Corollary 2.9 Let $G$ and $H$ be connected nontrivial graphs. Suppose that $H$ has a unique vertex that dominates $V(H)$. Then
\((i) \gamma \gamma_i (G[H]) = \gamma \gamma_i (G) \) if and only if \( G \) has an \( \gamma \gamma_i \)-pair \((A_0, B_0)\) such that \(|A_0| + |B_0| \leq |A| + |B| + |A^c \cap B| (\gamma'(H) - 1)\) for all dominating sets \( A \) in \( G \) and independent dominating sets \( B \) in \( G \).

\((ii) \) If \( \gamma_i (G) = \gamma (G) \), then \( \gamma \gamma_i (G[H]) = 2 \gamma (G) \).

**Example 2.10** Let \( G \) be any connected nontrivial graph. Then

\[(i) \gamma_i (G[K_{1,n}]) = \min \{|A| + (n-1)|A^c \cap B|: A \in \mathcal{D}(G), B \in \mathcal{J}(G)\}, |B| = \gamma_i (G) \] and

\[(ii) \gamma_i (G[K_p]) = \gamma (G) \text{ and } \gamma \gamma_i (G[K_p]) = 2 \gamma (G) \text{ for } p \geq 2.\]

**Proposition 2.11** For noncomplete connected graphs \( G \) and \( p \geq 2 \),

\[
\gamma_i (K_p [G]) = \begin{cases} 1, & \text{if } \gamma (G) = 1, \\ 2, & \text{otherwise.} \end{cases}
\]

**Proof:** If \( \gamma (G) = 1 \), then \( \gamma_i (K_p [G]) = \gamma (K_p) = 1 \), by Theorem 2.6(i) and Corollary 2.9(ii). Suppose that \( \gamma (G) \geq 2 \). Then \( \gamma (K_p [G]) \geq 2 \) and, hence, \( \gamma_i (K_p [G]) \geq 2 \). Let \( D \subseteq V(K_p [G]) \) be a \( \gamma_i \)-set in \( K_p [G] \). By Corollary 2.3, \( D = \bigcup_{i \in A} \{ (u, v) \} \) for some \( \gamma_i \)-set \( A \) in \( G \) and \( u \in V(K_p) \). Since \( G \) is nontrivial, \( V(G) \backslash A \neq \emptyset \). Let \( y \in V(G) \backslash A \) and put \( S = \{ (u, y), (x, y) \} \), where \( u \) and \( x \) are distinct vertices of \( K_p \). Since \( S \) is a dominating set in \( K_p [G] \) and \( S \cap D = \emptyset \), \( S \) is an inverse independent dominating set in \( K_p [G] \). Thus \( \gamma_i (K_p [G]) = 2 \).

**Corollary 2.12** For noncomplete connected graphs \( G \) and \( p \geq 2 \),

\[
\gamma \gamma_i (K_p [G]) = \begin{cases} 2, & \text{if } \gamma (G) = 1, \\ 4, & \text{otherwise.} \end{cases}
\]

**Proof:** If \( \gamma (G) = 1 \), then Inequality 2 yields \( \gamma \gamma_i (K_p [G]) = 2 \). Suppose that \( \gamma (G) \geq 2 \). Then \( 2 \leq \gamma \gamma_i (K_p [G]) \leq 4 \).

\( \gamma \gamma_i (K_p [G]) = 3 \), then \( \gamma (G) = 1 \), a contradiction. Thus, \( \gamma \gamma_i (K_p [G]) = 4 \).

**Theorem 2.13** [6] Let \( G \) and \( H \) be connected graphs. Then \( \mathcal{C} = \bigcup_{xy \in S} \{ \{ x \} \times T_x \} \subseteq V(G[H]), \) where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for each \( x \in S \), is a total dominating set in \( G[H] \) if and only if

\((i) \) \( S \) is a total dominating set in \( G \) or

\((ii) \) \( S \) is a dominating set in \( G \) and \( T_x \) is a total dominating set in \( H \) for every \( x \in S \backslash \mathcal{N}_G (S) \).

**Theorem 2.14** Let \( G \) and \( H \) be connected nontrivial graphs and \( \gamma (H) = 1 \). Then

\((i) \) if \( H \) has (at least) two distinct vertices each of which dominates \( V(H) \), then \( \gamma \gamma_i (G[H]) = \gamma (G) + \gamma_i (G); \)

\((ii) \) if \( H \) has a unique vertex that dominates \( V(H) \), then

\[
\gamma \gamma_i (G[H]) = \min \{|A| + |B|: A \in \mathcal{D}(G), B \in \mathcal{J}(G)\}.
\]

**Proof:** (i) Suppose that \( H \) has distinct vertices \( u \) and \( v \) such that \( \mathcal{N}_H [u] = V(H) \backslash \mathcal{N}_H [v] \). Let \( S, S' \subseteq V(G) \) be a \( \gamma \)-set and a \( \gamma_i \)-set in \( G \). Define \( D = S \backslash \{ u \} \) and \( T = S' \backslash \{ v \} \). Then \( (S, T) \) is a \( dt \)-pair in \( G[H] \). Thus,

\[
\gamma (G) + \gamma_i (G) = \gamma (G[H]) + \gamma_i (G[H]) \leq \gamma \gamma_i (G[H]) \leq |D| + |T| = \gamma (G) + \gamma_i (G),
\]

and \( \gamma (G) + \gamma_i (G) = \gamma (G) + \gamma_i (G) \).
(ii) Define \( \alpha = \min\{|A| + |B|: A \in \mathcal{D}(G), B \in \mathcal{T}(G)\} \). Let \( A \in \mathcal{D}(G) \) and \( B \in \mathcal{T}(G) \). Let \( v \in V(H) \) be such that \( N_H[v] = V(H) \) and \( w \in V(H) \setminus \{v\} \). Then \( (D; T) \), where \( D = A \times \{v\} \) and \( T = B \times \{w\} \), is a \( dt \)-pair in \( G[H] \). This means that \( \gamma(G[H]) \leq |D| + |T| = |A| + |B| \). Since \( A \) and \( B \) are arbitrary, \( \gamma(G[H]) \leq \alpha \).

Let \( (D; T) \) be a \( \gamma', \gamma \)-pair in \( G[H] \). Then \( D = \bigcup_{v \in E} \{(u) \times T_u\} \) and \( T = \bigcup_{u \in B} \{(u) \times T_u\} \) for some dominating sets \( A \) and \( B \) in \( G \) and \( T_u \subseteq V(H) \). Moreover, by Theorem 2.1, if \( A \) is not a total dominating set in \( G \), then \( T_u \) is a dominating set for all \( u \in A^+ \). Also, by Theorem 2.13, if \( B \) is not a total dominating set in \( G \), then \( T_u \) is a total dominating set in \( H \) for all \( u \in B^+ \). If \( B \) is a total dominating set in \( G \), then

\[ |D| + |T| \geq |A| + |B| \geq \alpha. \]

Suppose that \( B \) is not a total dominating set in \( G \). For each \( u \in B^+ \), if \( T_u = \{v \in V(H): (u, v) \in T\} \), then \( |T_u| \geq 2 \) so that

\[ |\bigcup_{u \in B^+} \{(u) \times T_u\}| \geq 2|B^+. \]

For each \( u \in B^+ \), choose \( v(u) \in V(G) \) such that \( uv(u) \in E(G) \). Define \( C = (B \setminus B^+) \cup B^+ \cup \{v(u): u \in B^+\} \). Then \( C \) is a total dominating set in \( G \) and

\[ |D| + |T| \geq |A| + |C| \geq \alpha. \]

This completely proves the Statement (ii) of the theorem.

3. REFERENCES