ON $g^p$-CONTINUOUS MAPS IN TOPOLOGICAL SPACES

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Abstract: The aim of this paper is to introduce and study $g^p$-continuous maps. Basic characterizations and several properties concerning them are obtained. Further, $g^p$- irresolute map is also defined. Some of the properties are investigated.

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1. INTRODUCTION

N. Levine [15] introduced the class of g-closed sets. M.K.R.S. Veerakumar introduced several generalized closed sets namely, g*-closed sets, g*-p-closed sets, g* p-continuous maps, g* p- irresolute maps and their properties. The authors have already introduced $g^p$-continuous sets and their properties. In this paper we study the new class of map called as $g^p$-continuous maps. Different characterizations of the introduced concepts are also found. In this direction $g^p$- irresolute maps are defined and some of their properties are studied by giving some counter example.

2. PRELIMINARIES

Throughout this paper (X, τ) (or X) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and C(A) denote the closure of A, the interior of A and the complement of A respectively.

We recall the following definitions which are useful in the sequel.

**Definition 2.1:** A subset A of a space (X, τ) is called
(i) a semi-open set [16] if $A \subseteq \text{Cl}(\text{Int}(A))$ and a semi-closed [16] if $\text{Int}(	ext{Cl}(A)) \subseteq A$.
(ii) a preopen set [20] if $A \subseteq \text{Int}(\text{Cl}(A))$ and a preclosed [20] if $\text{Cl}(\text{Int}(A)) \subseteq A$.
(iii) an α-open set [22] if $A \subseteq \text{Int}(\text{Cl}(A))$ and an α-closed [22] if $\text{Cl}(\text{Int}(A)) \subseteq A$.
(iv) a semi-preopen set [4] (β-open [1]) if $A \subseteq \text{Cl}(\text{Int}(A))$ and a semi-preclosed set [4] (β-closed [1]) if $\text{Int}(\text{Cl}(A)) \subseteq A$ and
(v) a regular open [14] if $\text{Int}(A) = \text{Int}(\text{Cl}(A))$ and a regular closed [14] if $\text{Cl}(\text{Int}(A)) = A$.

The semi-closure (resp. preclosure, α-closure, semi-preclosure) of a subset A of a space (X, τ) is the intersection of all semi-closed (resp. preclosed, α-closed, and semi-preclosed) sets that contain A and is denoted by scl(A) (resp. pcl(A), αcl(A) and spcl(A)).

**Definition 2.2:** A subset A of a space (X, τ) is called
(i) a generalized closed (briefly g-closed) set [15] if cl(A) U whenever A U and U is open in (X, τ).
(iii) a generalized semi-closed (briefly gs-closed) set [6] if scl(A) U whenever A U and U is open in (X, τ).
(iv) an α-generalized closed (briefly αg-closed) set [17] if αcl(A) U whenever A U and U is open in (X, τ).
(v) a generalized α-closed (briefly gα-closed) set [18] if αcl(A) U whenever A U and U is α-open in (X, τ). The complement of a gα-closed set is called a gα-open [9] set.
(vi) a **generalized preclosed** (briefly gp-closed) set [19] if \( \text{pcl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

(vii) a **generalized semi-closed** (briefly gsp-closed) set [11] if \( \text{spcl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

(viii) a **generalized preregular closed** (briefly gpr-closed) set [12] if \( \text{pcl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is regular open in \((X, \tau)\).

(ix) a **g\#-closed** set [27] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \alpha \)-g- open in \((X, \tau)\).

(x) a **g\#-closed** set [28] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( g \)-open in \((X, \tau)\). The complement of a \( g^\# \)-closed set is called a **\( g^\# \)-open** [28] set.

(xi) a **g\#-pre closed set** [25] (briefly \( g^\#p \)-closed) set if \( \text{pcl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a \( g^\# \)-open set in \((X, \tau)\).

(xii) a **g\#-closed** set [2] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( g^\# \)-open in \((X, \tau)\).

**Definition 2.3:** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be

(i) **semi-continuous** [16] if \( f^{-1}(V) \) is semi-open in \((X, \tau)\) for every open set \( V \) of \((Y, \sigma)\).

(ii) **pre-continuous** [20] if \( f^{-1}(V) \) is preclosed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

(iii) **\( \alpha \)-continuous** [21] if \( f^{-1}(V) \) is \( \alpha \)-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

(iv) **\( \beta \)-continuous** [1] if \( f^{-1}(V) \) is semi-preopen in \((X, \tau)\) for every open set \( V \) of \((Y, \sigma)\).

(v) **\( g \)-continuous** [7] if \( f^{-1}(V) \) is \( g \)-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

(vi) **sg-continuous** [24] if \( f^{-1}(V) \) is sg-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

(vii) **gs-continuous** [10] if \( f^{-1}(V) \) is gs-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

(viii) **\( g\alpha \)-continuous** [18] if \( f^{-1}(V) \) is \( g\alpha \)-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

(ix) **\( g\beta \)-continuous** [12] if \( f^{-1}(V) \) is \( g\beta \)-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

(x) **gsp-continuous** [11] if \( f^{-1}(V) \) is gsp-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

(xi) **gp-continuous** [23] if \( f^{-1}(V) \) is gp-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

(xii) **gpr-continuous** [12] if \( f^{-1}(V) \) is gpr-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

(xiii) **gp-irresolute** [5] if \( f^{-1}(V) \) is gp-closed in \((X, \tau)\) for every gp-closed set \( V \) of \((Y, \sigma)\).

(xiv) **p-open** [13] if \( f(U) \) is preopen in \((Y, \sigma)\) for every preopen set \( U \) in \((X, \tau)\).

(xv) **pre-\( \alpha \)-open** [9] if \( f(U) \) is \( \alpha \)-closed in \((Y, \sigma)\) for every \( \alpha \)-closed set \( U \) in \((X, \tau)\).

(xvi) **\( g^\# \)-continuous** [27] if \( f^{-1}(V) \) is \( g^\# \)-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

(xvii) **\( g^\# \)-irresolute** [27] if \( f^{-1}(V) \) is \( g^\# \)-closed in \((X, \tau)\) for every \( g^\# \)-closed set \( V \) of \((Y, \sigma)\).

(xviii) **\( g^\# \)-continuous** [28] if \( f^{-1}(V) \) is \( g^\# \)-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

(xix) **\( g^\#p \)-continuous** [3] if \( f^{-1}(V) \) is \( g^\#p \)-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

**Definition 2.4:** A space \((X, \tau)\) is called a

(i) **\( T_{gp}^\# \) space** [26] if every \( g^\#p \)-closed set is closed.

(ii) **\( T_{g^\#p} \) space** [26] if every gp-closed set is \( g^\#p \)-closed.

(iii) **\( T_{gp}^\# \) space** [26] if every \( g^\#p \)-closed set is \( g \)-closed.

(iv) **\( T_{gp}^\# \) space** [26] if every \( g^\#p \)-closed set is preplosed.

(v) **\( T_{gp}^\# \) space** [26] if every \( g^\#p \)-closed set is \( \alpha \)-closed.
3. \(g^p\) -CONTINUOUS MAPS AND \(g^p\) -IRRESOLUTE MAPS

We introduce the following definition

**Definition 3.1:** A function \(f:(X,\tau)\to(Y,\sigma)\) is said to be \(g^p\)-continuous if \(f^{-1}(V)\) is a \(g^p\)-closed set of \((X,\tau)\) for every closed set \(V\) of \((Y,\sigma)\).

**Theorem 3.2:**

(i) Every continuous map is \(g^p\)-continuous .

(ii) Every \(g^p\)-continuous map is \(g^p\)-continuous .

(iii) Every \(g^p\)-continuous map is \(g^p\)-continuous .

(iv) Every \(g^p\)-continuous map is \(ag\)-continuous, \(gs\)-continuous, \(gspr\)-continuous, \(g^p\)-continuous and \(gpr\)-continuous .

**Proof:**
Follows from the theorem 3.2 [2]

The converse of the theorem 3.2 need not be true as can be seen from the following examples.

**Example 3.3:** Let \(X=\{a,b,c\}=Y,\tau=\{\emptyset,X,\{a,b\}\}\) and \(\sigma=\{\emptyset,Y,\{a\}\}\). Define \(f:(X,\tau)\to(Y,\sigma)\) by \(f(a)=b,f(b)=a\) and \(f(c)=c\). \(f\) is not continuous map, since \(\{a\}\) is an open set of \((Y,\sigma)\) but \(f^{-1}(\{a\})=\emptyset\) is not an open set of \((X,\tau)\). But it is \(g^p\)-continuous map.

**Example 3.4:** Let \(X=\{a,b,c\}=Y,\tau=\{\emptyset,X,\{a,b\}\}\) and \(\sigma=\{\emptyset,Y,\{a\}\}\). Define \(g:(X,\tau)\to(Y,\sigma)\) by \(g(a)=b,g(b)=c\) and \(g(c)=a\). \(g\) is \(g^p\)-continuous map. But not \(g\)-continuous, since \(\{b\}\) is a closed set of \((Y,\sigma)\) but \(g^{-1}(\{b\})=\{a\}\) is not a \(g\)-closed set of \((X,\tau)\). Also it is not \(g^p\)-continuous map.

**Example 3.5:** Let \(X=\{a,b,c\}=Y,\tau=\{\emptyset,X,\{a\}\}\) and \(\sigma=\{\emptyset,Y,\{a\}\}\). Define \(h:(X,\sigma)\to(Y,\sigma)\) by \(h(a)=c,h(b)=b\) and \(h(c)=a\). \(h\) is not \(g^p\)-continuous map, since \(\{c\}\) is a closed set of \((Y,\sigma)\) but \(h^{-1}(\{c\})=\{a\}\) is not a \(g^p\)-closed set of \((X,\tau)\). But it is \(ag\)-continuous, \(gs\)-continuous, \(gspr\)-continuous, \(g^p\)-continuous and \(gpr\)-continuous.

Thus the class of \(g^p\)-continuous maps properly contains the classes of continuous maps, \(g\)-continuous maps and \(g^p\)-continuous maps. Next we show that the class of \(g^p\)-continuous maps is properly contained in the classes of \(ag\)-continuous, \(gs\)-continuous, \(gspr\)-continuous, \(g^p\)-continuous and \(gpr\)-continuous.

**Theorem 3.6:**

(i) \(g^p\)-continuity is independent of semi-continuity and \(\beta\)-continuity.

(ii) \(g^p\)-continuity is independent of \(sg\)-continuity, \(a\)-continuity, \(pg\)-continuity, \(pre\)-continuity, and \(gs\)-continuity

**Example 3.7:** Let \(X=\{a,b,c\}=Y,\tau=\{\emptyset,X,\{a\}\}\). Define \(f:(X,\tau)\to(Y,\sigma)\) by \(f(a)=b,f(b)=c\) and \(f(c)=a\). \(f\) is \(g^p\)-continuous map. But not \(\beta\)-continuity and semi-continuity maps, since \(\{b,c\}\) is a closed set of \((Y,\sigma)\) but \(f^{-1}(\{b,c\})=\{a,b\}\) is not semipreclosed and semiclosed set.

**Example 3.8:** Let \(X=\{a,b,c\}=Y,\tau=\{\emptyset,X,\{a\}\}\) and \(\sigma=\{\emptyset,Y,\{a\}\}\). Define \(h:(X,\tau)\to(Y,\sigma)\) by \(h(a)=c,h(b)=b\) and \(h(c)=a\). \(h\) is not \(g^p\)-continuous map, since \(\{c\}\) is a closed set of \((Y,\sigma)\) but \(h^{-1}(\{c\})=\{a\}\) is not a \(g^p\)-closed set of \((X,\tau)\). But it is semi-continuous and \(\beta\)-continuous map.

**Example 3.9:** Let \(X=\{a,b,c\}=Y,\tau=\{\emptyset,X,\{a,c\}\}\) and \(\sigma=\{\emptyset,Y,\{a,b\}\}\). Define \(0:(X,\tau)\to(Y,\sigma)\) by \(0(a)=a,0(b)=c,0(c)=b\). \(0\) is a \(g^p\)-continuous map. But it is not \(sg\)-continuous, \(a\)-continuous, \(pg\)-continuous, \(pre\)-continuous and \(gs\)-continuous, since \(\{a,b\}\) is a closed set of \((Y,\sigma)\) but \(0^{-1}(\{a,b\})=\{a,c\}\) is not \(sg\)-closed, \(a\)-closed, \(pg\)-closed, \(pre\)-closed and \(gs\)-closed sets.

**Example 3.10:** Let \(X,Y,\tau\) be as in the example 3.9 and \(\sigma=\{\emptyset,Y,\{a\}\}\). Define \(h:(X,\tau)\to(Y,\sigma)\) by \(h(a)=c,h(b)=b\) and \(h(c)=a\). \(h\) is not \(g^p\)-continuous map, since \(\{b\}\) is a closed set of \((Y,\sigma)\) but \(h^{-1}(\{b\})=\{b\}\) is not a \(g^p\)-closed set of \((X,\tau)\). But it is \(sg\)-continuous, \(a\)-continuous, \(pg\)-continuous, \(pre\)-continuous and \(gs\)-continuous maps.

**Remark 3.11:** Composition of two \(g^p\)-continuous maps need not be \(g^p\)-continuous maps as it can be seen from the following example.

**Example 3.12:** Let \(X=\{a,b,c\}=Y=Z,\tau=\{\emptyset,X,\{a\}\}\) and \(\sigma=\{\emptyset,Y,\{a\}\}\) and \(\eta=\{\emptyset,Z,\{a\}\}\). Define \(f:(X,\tau)\to(Y,\sigma)\) by \(f(a)=b,f(b)=c,f(c)=a\). Define \(g:(Y,\sigma)\to(Z,\eta)\) by \(g(a)=b,g(b)=c,g(c)=a\). Clearly \(f\) and \(g\) are \(g^p\)-continuous maps. But \(g o f:(X,\tau)\to(Z,\eta)\) is not \(g^p\)-continuous, since \(\{b,c\}\) is a closed set of \((Z,\eta)\) but \((g o f)^{-1}(\{b,c\})=f^{-1}(g^{-1}(\{b,c\})=f^{-1}(\{a\})=\{c,a\}\) is not a \(g^p\)-closed set of \((X,\tau)\).
**Definition 3.13:** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $g^p$-irresolute if $f^{-1}(V)$ is a $g^p$-closed set of $(X, \tau)$ for every $g^p$-closed set $V$ of $(Y, \sigma)$.

**Theorem 3.14:** Every $g^p$-irresolute map is $g^p$-continuous.

**Proof:** The first assertion follows from the fact that every closed set is a $g^p$-closed set.

The converse of the theorem 3.14 is not true as it can be seen by the following example.

**Example 3.15:** Let $X = \{a, b, c\} = Y, \tau = \{\emptyset, X, \{a\}, \{a, c\}\}$ and $\sigma = \{\emptyset, Y, \{a\}\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = a, f(b) = c$ and $f(c) = b$. $f$ is not $g^p$-irresolute, since $\{b\}$ is a $g^p$-closed set of $(Y, \sigma)$ but $f^{-1}(\{c\}) = \{a\}$ is not a $g^p$-closed set of $(X, \tau)$. But it is $g^p$-continuous.

**Theorem 3.16:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be any two functions. Then

(i) $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is $g^p$-continuous if $g$ is continuous and $f$ is $g^p$-continuous.

(ii) $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is $g^p$-irresolute if $g$ is $g^p$-irresolute and $f$ is $g^p$-continuous.

(iii) $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is $g^p$-continuous if $g$ is $g^p$-continuous and $f$ is $g^p$-irresolute.

The proof is obvious from the definitions 3.1 and 3.13.

**Theorem 3.17:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a bijective $g^p$-irresolute and closed map. Then $f(A)$ is a $g^p$-closed set in $(Y, \sigma)$ for every $g^p$-closed set $A$ of $(X, \tau)$.

**Proof:** Let $A$ be a $g^p$-closed set of $(X, \tau)$. Let $V$ be a $g^p$-open set of $(Y, \sigma)$ containing $f(A)$. Since $f$ is $g^p$-irresolute, then $f^{-1}(V)$ is a $g^p$-open set of $(X, \tau)$. Since $A \subseteq f^{-1}(V)$ and $A$ is $g^p$-closed, then $cl(A) \subseteq f^{-1}(V)$. Then $f(cl(A)) \subseteq V$. Hence $f(A)$ is a $g^p$-closed set in $(Y, \sigma)$.

**Theorem 3.18:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $g^p$-continuous map.

(i) If $(X, \tau)$ is a $T^p_1$ space, then $f$ is continuous.

(ii) If $(X, \tau)$ is a $T^p_2$ space, then $f$ is $g\alpha$-continuous.

(iii) If $(X, \tau)$ is a $T^p_3$ space, then $f$ is pre-continuous.

(iv) If $(X, \tau)$ is a $T^p_4$ space, then $f$ is $\alpha$-continuous.

**Theorem 3.19:** Let $(X, \tau) \rightarrow (Y, \sigma)$ be a $g^p$-continuous map. If $(X, \tau)$ is a $T^p_{\alpha}$ space, then $f$ is $g^p$-continuous.

**Theorem 3.20:** Let $(X, \tau) \rightarrow (Y, \sigma)$ be a $g^p$-continuous map. If $(X, \tau)$ is a $T^p_{\alpha}$ space, then $f$ is $g^p$-continuous.

**Theorem 3.21:** Let $(X, \tau) \rightarrow (Y, \sigma)$ be onto, $g^p$-irresolute and closed. If $(X, \tau)$ is a $T^p_{\alpha}$ space, then $(Y, \sigma)$ is also a $T^p_{\alpha}$ space.

**Proof:** Let $A$ be a $g^p$-closed set of $(Y, \sigma)$. Since $f$ is $g^p$-irresolute, then $f^{-1}(A)$ is a $g^p$-closed set of $(X, \tau)$. Since $(X, \tau)$ is a $T^p_{\alpha}$ space, then $f^{-1}(A)$ is closed in $(X, \tau)$. Since $f$ is closed and onto, then $A = f(f^{-1}(A))$ is closed in $(Y, \sigma)$. Hence $(Y, \sigma)$ is also a $T^p_{\alpha}$ space.

**Definition 3.22:** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be pre-$g^p$-closed if $f(U)$ is $g^p$-closed in $(Y, \sigma)$ for every $g^p$-closed set $U$ in $(X, \tau)$.

**Definition 3.23:** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be pre-$g\alpha$-closed if $f(U)$ is $g\alpha$-closed in $(Y, \sigma)$ for every $g\alpha$-closed set $U$ in $(X, \tau)$.

**Theorem 3.24:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be onto, $g^p$-irresolute and pre-$g\alpha$-closed. If $(X, \tau)$ is a $T^p_{\alpha}$ space, then $(Y, \sigma)$ is also a $T^p_{\alpha}$ space.

**Theorem 3.25:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be onto, $g^p$-irresolute and $p$-closed. If $(X, \tau)$ is an $aT^p_{\alpha}$ space, then $(Y, \sigma)$ is also an $aT^p_{\alpha}$ space.

**Theorem 3.26:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be onto, $g^p$-irresolute and pre-$\alpha$-closed. If $(X, \tau)$ is an $aT^p_{\alpha}$ space, then $(Y, \sigma)$ is also an $aT^p_{\alpha}$ space.

**Theorem 3.27:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be onto, $g^p$-irresolute and pre-$g^p$-closed. If $(X, \tau)$ is a $T^p_{\alpha}$ space, then $(Y, \sigma)$ is also a $T^p_{\alpha}$ space.

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