EXISTENCE OF UNIFORM GLOBAL LOCALLY ATTRACTIVE SOLUTIONS
FOR FRACTIONAL ORDER NONLINEAR RANDOM INTEGRAL EQUATION

B.D. Karande
Department of Mathematics,
Maharashtra Udayagiri Mahavidyalaya, Udgir-413517, (M.S) INDIA
E-mail: bdkarande@rediffmail.com

Abstract: In this Paper, we discuss the existence of solutions for fractional order nonlinear random integral equation in set of nonnegative real numbers under lipshitz and Caratheodory conditions. Moreover, we show that solutions of this equation are uniformly locally attractive solutions is proved.

Keywords: Banach algebra, fractional order nonlinear random integral Sequation, existence result, locally attractive solution.

INTRODUCTION

In the theory of diffusion or heat-condition we have the diffusio coefficients or the coefficients of conductivity play the prominent role in the said phenomena. Similarly, in the wave theory, the propagation coefficients and in the theory of elasticity, the modules of elasticity play the significant role in the behavior of the underlined processes. The coefficients or parameters that have an important role in the natural processes are called random parameters. Hence when we talk about some parameters or coefficients, the random analysis of the random equations is evident. Therefore the random equations have been studied in the literature, since long time, by various Mathematicians all over the world. Thus the study of a natural or physical phenomenon with the help of random models or equations forms an important branch of the analysis. The theory of differential and integral equations of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs devoted to differential and integral equations of fractional order have appeared ([1, 3, 14-21]). These papers contain various types of existence results for equations of fractional order. In this paper we study the existence of locally attractive solution for fractional order nonlinear random integral equation in Banach Algebra.

By a random solution of the FRIE (1.1) we mean a function \( x: \Omega \rightarrow BC \) that satisfies (1.1) on \( \Omega \). Where \( BC \) is the space of bounded continuous real valued functions defined on \( \Omega \). Consider the fractional order nonlinear functional random integral equation (in short FRIE) of mixed type

\[
x(t, \omega) = f(t, x(t, \theta), \eta, \omega) + \frac{1}{\Gamma(\alpha)} \int_0^t g(t, x(s, \eta), \omega, \omega, \omega) \frac{ds}{(t-s)^{1-\alpha}}
\]

for all \( t \in \mathbb{R}_+, \alpha \in 0,1 \) and \( \omega \in \Omega \), where \( q: \mathbb{R}_+ \times \Omega \rightarrow C \), \( f, g: \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R} \), \( \theta, \eta, \mu: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \).

PRELIMINARIES

Let \( X = BC \) be Banach algebra with norm \( \| \cdot \| \) and let \( \Omega \) be a subset of \( X \). Let a mapping \( A: X \rightarrow X \) be an operator and consider the following operator equation in \( X \), namely,
\[ x_t = Ax_t \]

for all \( t \in \mathbb{R}_+ \). Below we give different characterizations of the solutions for operator equation (2.1) on \( \mathbb{R}_+ \). We need the following definitions in the sequel See [20,21].

**Definition 2.1:** We say that solutions of the equation (2.1) are locally attractive if there exists an \( x_0 \in BC \mathbb{R}_+,\mathbb{R} \) and an \( r > 0 \) such that for all solutions \( x = x_t \) and \( y = y_t \) of equation (2.1) belonging to \( B_r x_0 \cap \Omega \) we have that

\[
\lim_{t \to \infty} x_t - y_t = 0 \quad (2.2)
\]

In case the limit (2.2) is uniform with respect to the set \( B_r \cap \Omega \), i.e. when for each \( \varepsilon > 0 \) there exist \( T > 0 \) such that

\[
| x_t - y_t | \leq \varepsilon \quad (2.3)
\]

for all \( x, y \in B_r \cap \Omega \) being solutions of (2.1) and for \( t \geq T \), then we say that solution of equation (2.1) are uniformly locally attractive (or equivalently, that solutions of (2.1) are asymptotically stable).

**Definition 2.2:** The solution \( x = x_t \) of equation (2.1) is said to be globally attractive if (2.2) holds for other solution \( y = y_t \) of (2.1) in \( \Omega \). In other words, we may say that solutions of equation (2.1) are globally attractive if for arbitrary solution \( x_t \) and \( y_t \) of (2.1) on \( \Omega \), the condition (2.2) is satisfied. In the case when the condition (2.2) is satisfied uniformly with respect to the set \( \Omega \), i.e. if for every \( \varepsilon > 0 \) there exist \( T > 0 \) such that the inequality (2.3) is satisfied for all \( x, y \in \Omega \) being solutions of (2.1) and for \( t \geq T \), then we say that solution of equation (2.1) are uniformly globally attractive on \( \mathbb{R}_+ \).

**Definition 2.3:** A line \( y = at + b \), where \( a \) and \( b \) are real numbers, is called an attractor for the solution \( x \in BC \mathbb{R}_+,\mathbb{R} \) to equation (2.5) if \( \lim_{t \to \infty} [ x_t - at - b ] = 0 \). In this case the solution \( x \) to equation (2.1) is also called to be asymptotic to the line \( y = at + b \) and the line is an asymptote for the solution \( x \) on \( \mathbb{R}_+ \).

**Definition 2.4:** The solutions of equation (2.1) are said to be globally asymptotically attractive if for any two solutions \( x = x_t \) and \( y = y_t \) of equation (2.1), the condition (2.2) is satisfied and there is a line which is a common attractor to them on \( \mathbb{R}_+ \). If condition (2.2) is satisfied uniformly, i.e., if for every \( \varepsilon > 0 \) there exists \( T > 0 \) such that the inequality (2.3) is satisfied for \( t \geq T \) for all \( x \) and \( y \) being the solutions of (2.1) and have a line as a common attractor, then we say that solutions of equations of equation (2.1) are uniformly globally asymptotically attractive on \( \mathbb{R}_+ \).

**Remark:** Let us mention that the global attractivity of solutions is recently introduced in BapuraoC.Dhage[12]

**Definition 2.5:** Let \( X \) be a Banach space. A mapping \( A : X \to X \) is called **Lipschitz** if there is a constant \( \alpha > 0 \) such that

\[
\| Ax - Ay \| \leq \alpha \| x - y \| \quad \text{for all } x, y \in X.
\]

If \( \alpha < 1 \), then \( A \) is called a **contraction** on \( X \) with the contraction constant \( \alpha \).

**Definition 2.6:** (Dugundji and Granas [13]). An operator \( A \) on a Banach space \( X \) into itself is called **Compact** if for any bounded subset \( S \) of \( X \), \( A(S) \) is a relatively compact subset of \( X \). If \( A \) is continuous and compact, then it is called completely continuous on \( X \).
We seek the solutions of (1.1) in the space $BC_{+}$ of continuous and bounded real-valued functions defined on $\mathbb{R}_+$. Define a standard supremum norm $\|\cdot\|$ and a multiplication "$.$" in $BC_{+}$ by $\|x\| = \sup_{t \in \mathbb{R}_+} |x(t)|$.

$xy \cdot t = x(t) \cdot y(t), t \in \mathbb{R}_+.$ 2.4

Clear, $BC_{+}$ becomes a Banach space with respect to the above norm and the multiplication in it. By $L^1_{+}$ we denote the space of Lebesgue integrable functions on $\mathbb{R}_+$ with the norm $\|\cdot\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^\infty |x(t)| dt.$$ 2.5

Denote by $L^1_{a,b}$ be the space of Lebesgue integrable functions on the interval $(a, b)$, which is equipped with the standard norm. Let $x \in L^1_{a,b}$ and let $\beta > 0$ be a fixed number.

**Definition 2.7:** The left sided Riemann-Liouville fractional integral of order $\beta$ of real function $f$ is defined as

$$I^\beta_\alpha f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(t)}{(x-t)^{1-\beta}} dt, \quad \beta > 0, x > a$$

**Definition 2.8:** The Riemann-Liouville fractional integral of order $\beta$ of the function $x(t)$ is defined by the formula:

$$I^\beta x(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{x(s)}{(t-s)^{1-\beta}} ds, \quad t \in (a,b)$$

Where $\Gamma(\beta)$ denote the gamma function.

It may be shown that the fractional integral operator $I^\beta$ transforms the space $L^1_{a,b}$ into itself and has some other properties (see [18-20]).

**Theorem 2.1:** (Arzela-Ascoli theorem) If every uniformly bounded and equi-continuous sequence $\{f_n\}$ of functions in $C(\mathbb{R}_+)$, then it has a convergent subsequence.

**Theorem 2.2:** A metric space $X$ is compact iff every sequence in $X$ has a convergent subsequence.

**Theorem 2.3:** Let $S$ be a closed convex and bounded subset of the Separable Banach space $X$ and let $A\omega, B\omega : \Omega \times S \to X$ be two operators satisfying for each $\omega \in \Omega$:

(a) $A\omega$ is $\mathcal{D}$-Lipschitzian ,

(b) $B\omega$ is completely continuous, and

(c) $A\omega x B\omega x \in S$ for all $x \in S$ ,and

Then the operator equation $AxBy = x$ has a random solution whenever $M\omega \phi_\omega < r, r > 0$, for each $\omega \in \Omega$ where $M\omega = \|B\omega \| = \sup \|Bx\| : x \in S$. 

**EXISTENCE RESULT**

**Definition 3.1.** A mapping $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is said to be $L^1_{\omega}$-Caratheodory if

1. $t \mapsto \beta(t,x,\omega)$ is measurable for all $\omega \in \Omega$ ,

2. $x \mapsto \beta(t,x,\omega)$ is continuous almost everywhere for $t \in \mathbb{R}_+$ and

3. for each real number $r > 0$ there exists a function $h_r : \Omega \to L^1_{+}$ such that $|\beta(t,x)| \leq h_r(t,\omega)$ a.e.

$t \in \mathbb{R}_+$ for all $x \in \Omega$ with $|x| \leq r$.

For convenience, the function $h$ is referred to as a **bound function** of $g$. 

---

© JGRMA 2012. All Rights Reserved
We consider the following hypotheses in the sequel.

- **H$_1$** The function $\theta, \eta, \mu : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous.

- **H$_2$** The function $\omega \to f(t, x, \omega)$ is measurable for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

- **H$_3$** The function $f : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is continuous and bounded with bound $\phi_{\omega} = \sup_{t, x, \omega \in \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega} |f(t, x, \omega)|$ there exists a bounded function $\alpha_{t} : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ with bound $\|\alpha_{t}\|$ such that for each $\omega \in \Omega$

  $|f(t, x, \omega) - f(t, y, \omega)| \leq \alpha_{t}(t, \omega) |x - y|$ a.e. $t \in \mathbb{R}_+$ for all $x, y \in \mathbb{R}$

- **B$_0$** The function $g : \mathbb{R}_+ \times \Omega \to C[\mathbb{R}_+, \mathbb{R}]$ is measurable, bounded and Vanish at infinity, that is $\lim_{t \to \infty} q(t, \omega) = 0$

- **B$_1$** The function $g$ is random $L^1_{\mu}$ Caratheodory.

- **B$_2$** There exists a continuous function $h : \Omega \to L^1_{\mu}$ such that $|g(t, x, \omega) - h(t, \omega)|$ a.e. $t \in \mathbb{R}_+$ for all $x \in \mathbb{R}$ and $\omega \in \Omega$. Moreover, we assume that $\lim_{t \to \infty} \mu(t, \omega) = 0$

In what follows assume additionally that the following conditions satisfied.

- **B$_3$** The function $a : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the formulas $a(t) = \|h_{t, \omega}(t, \omega)\|^{\alpha} t^\alpha$ is bounded on $\mathbb{R}_+$ and Vanish at infinity, that is $\lim_{t \to \infty} a(t) = 0$, there is a real number $T > 0$ such that $a(t) < \frac{\epsilon}{2}$ for all $t \geq T$.

**Remark 3.1**: Note that if the hypothesis $B_1 - B_3$ hold, then there exist constants $K_1 > 0$ and $K_2 > 0$ such that:

\[ K_1 = \sup_{q \in \mathbb{R}_+} q(t, \omega) : t \in [0, \omega] \in \Omega, K_2 = \sup_{\alpha \in [1, \infty]} \frac{\|h_{t, \omega}(t, \omega)\|^{\alpha}}{\Gamma(\alpha + 1)} : t \in [0, \omega] \in \Omega \]

**Theorem 3.1**: Assume that the hypotheses $H_1 - H_4$ hold. Further if $\phi_{\omega} K_1 + K_2 < 1$ where $K_1$ and $K_2$ are defined in Remark (3.1). Then the FRIE 1.1 has a random solution in the space $E = MB \times \Omega$ on $\mathbb{R}_+$. Moreover, random solutions are locally attractive on $\mathbb{R}_+$.

**Proof**: Let us denote $X = BM \times \Omega$ and $S$ is a nonempty subset of $X$ by

\[ S = \{ s \in X : \|s\| \leq M, \forall \omega \in \Omega \} \quad 3.1 \]

where $M \omega = L_1 + L_2$

Clearly $S$ is a closed convex and bounded subset of $X$. Define two operators $A(\omega), B(\omega) : \Omega \times S \to X$ by

\[ A(\omega)(x) = f(t, x, \theta, t, t, \omega, \omega) t \in \mathbb{R}_+, \quad 3.2 \]
for all \( t \in \mathbb{T}_+ \) and \( \omega \in \Omega \).

We show that \( A \) and \( B \) are random operators on \( \Omega \times S \). By hypothesis (H2), the map \( \omega \rightarrow \int t, x, \omega \) is measurable. Also by hypothesis (B1), the mapping \( \omega \rightarrow g \), \( t, x, \eta, s, \omega, \omega \) is measurable by Carathéodory theorem. Since the integral is a limit of the finite sum of measurable functions, we have the function \( \omega \rightarrow \frac{1}{\Gamma} \int_0^t g T \overset{\mu}{\underset{\nu}{\sum}} s, x, \eta, s, \omega, \omega \) is measurable.

Similarly the map \( \omega \rightarrow \int q \), \( t, \omega + \frac{1}{\Gamma} \int_0^t g T \overset{\mu}{\underset{\nu}{\sum}} s, x, \eta, s, \omega, \omega \) is measurable for all \( t \in \mathbb{T}_+ \). Consequently the map \( \omega \rightarrow A \omega \) and \( \omega \rightarrow B \omega \) are measurable for all \( x \in S \) and that \( A \) and \( B \) are the random operators on \( x \in S \).

Now the FRIE 1.1 is equivalent to the random equation \( A \omega \times B \omega = x \). We shall show that the random operators \( A \omega \) and \( B \omega \) satisfy all conditions of theorem 2.3 on \( S \).

**Step I:** First we show that \( A \omega \) is Lipschitzian, let \( x, y \in S \). Then by \( H_3 \)

\[
|A \omega \times t - A \omega \times y| = |f(t, x, \theta, t, \omega, \omega) - f(t, y, \theta, t, \omega, \omega)|
\]

\[
\leq \alpha_1 \omega \times |x - y| \quad 3.4
\]

Taking maximum overin the above inequality, we obtain

\[
\|A x - A y\| \leq \|\alpha_1 \omega\| \|x - y\| \text{ for all } x, y \in S \text{ and } \omega \in \Omega. \quad 3.5
\]

This shows that \( A \omega \) is a Lipschitzian random operator on \( S \) with Lipschitz constant \( \|\alpha_1 \omega\| \).

**Step II:** Next we will show the \( B \omega \) is completely continuous on \( S \).

**Case I:** let \( t \in [0, T] \) and let \( x_n \) be a sequence of points in \( S \) such that \( x_n \rightarrow x \) as \( n \rightarrow \infty \). Then, by dominated convergence theorem,

\[
\lim_{n \to \infty} B \omega \times x_n = \lim_{n \to \infty} \left( q t, \omega + \frac{1}{\Gamma} \int_0^t g T \overset{\mu}{\underset{\nu}{\sum}} s, x_n, \eta, s, \omega, \omega \right)
\]

\[
\leq q t, \omega + \lim_{n \to \infty} \left( \frac{1}{\Gamma} \int_0^t g T \overset{\mu}{\underset{\nu}{\sum}} s, x_n, \eta, s, \omega, \omega \right)
\]
\[ \leq q \ t, \omega + \left( \frac{1}{\Gamma \alpha} \int_0^\beta g_{s, x_n} \eta_{s, \omega}, \omega \ ds \right) \]

\[ \leq q \ t, \omega + \left( \frac{1}{\Gamma \alpha} \int_0^\beta g_{s, x} \eta_{s, \omega}, \omega \ ds \right) \]

\[ = B \ (t, \omega) \times t \text{ for all } t \in [0, T] \text{ and } \omega \in \Omega. \quad 3.6 \]

**Case II:** Suppose that \( t \geq T \). Then we have

\[ \left| B (t, \omega) - B (x, t) \right| \leq q (t, \omega) + \frac{1}{\Gamma \beta} \int_0^\beta \left| \frac{g_{s, x_n} \eta_{s, \omega}, \omega}{t - s} \right| ds - q (t, \omega) - \frac{1}{\Gamma \beta} \int_0^\beta \left| \frac{g_{s, x} \eta_{s, \omega}, \omega}{t - s} \right| ds \]

\[ \leq \frac{1}{\Gamma \alpha} \int_0^\beta \left| \frac{g_{s, x_n} \eta_{s, \omega}, \omega}{t - s} \right| ds - \frac{1}{\Gamma \alpha} \int_0^\beta \left| \frac{g_{s, x} \eta_{s, \omega}, \omega}{t - s} \right| ds \]

\[ \leq \frac{1}{\Gamma \alpha} \int_0^\beta \left| \frac{g_{s, x_n} \eta_{s, \omega}, \omega}{t - s} \right| ds + \frac{1}{\Gamma \alpha} \int_0^\beta \left| \frac{g_{s, x} \eta_{s, \omega}, \omega}{t - s} \right| ds \]

\[ \leq \frac{1}{\Gamma \alpha} \int_0^\beta \left| \frac{h_{s, \omega}}{t - s} \right| ds + \frac{1}{\Gamma \alpha} \int_0^\beta \frac{h_{s, \omega}}{t - s} ds \]

\[ \leq \frac{2}{\Gamma \alpha} \int_0^\beta \frac{h_{s, \omega}}{t - s} ds \leq \frac{2}{\Gamma \alpha} \left\| h_{M, \omega} \right\|_0 \left\| t - s \right\|_x \]

\[ \leq \frac{2}{\Gamma \alpha + 1} \left\| h_{M, \omega} \right\|_0 \left\| t - s \right\|_x \leq 2a \ t < \varepsilon \quad 3.7 \]

for all \( t \geq T \) and \( \omega \in \Omega \), since \( \varepsilon \) is arbitrary, one has \( \lim_{n \to \infty} B (t, \omega) x_n = B (t, \omega) x \) for all \( t \geq T \) and \( \omega \in \Omega \).

Now combining the case I with case II, we conclude that \( B (t, \omega) \) is continuous random operator on \( S \) into itself.

**Step III:** Next we show that \( B \) is compact on \( S \). Let \( x_n \) be a sequence in \( S \). Then \( \left\| x_n \right\| \leq M (\omega) \) for each \( n \in N \).

Since \( g \ t, x, \omega \) is \( L^1 \) Caratheodory,

\[ \left| B (t, \omega) x_n \right| \leq q (t, \omega) + \frac{1}{\Gamma \alpha} \int_0^\beta \left| \frac{g_{s, x_n} \eta_{s, \omega}, \omega}{t - s} \right| ds \]
\[
\begin{align*}
&\leq |q\ t,\omega| + \frac{1}{\Gamma\alpha} \int_0^t \lambda(t) ds - q\ t_1,\omega = \frac{1}{\Gamma\alpha} \int_0^t \lambda(t) ds - q\ t_1,\omega \\
&\leq \left|q\ t_2,\omega - q\ t_1,\omega\right| + \frac{1}{\Gamma\alpha} \int_{t_1}^{t_2} \lambda(t) ds
\end{align*}
\]

for all \( t \in R \) and \( \omega \in \Omega \). Taking supremum over \( t \) and \( \omega \), we obtain \( \|B\ \omega\ x_n\ t\| \leq K_1 + K_2 \) for all \( n \in N \). This shows that \( B\ \omega\ x_n \) is a uniformly bounded sequence in \( B\ \omega\ S \).

Now we proceed to show that the sequence \( B\ \omega\ x_n\ t \) is also equicontinuous in \( B\ \omega\ S \). Let \( t_1, t_2 \in \mathbb{R}_+ \) be arbitrary.

\[
\begin{align*}
&B\ \omega\ x_n\ t_2 - B\ \omega\ x_n\ t_1 \\
&\leq \left|q\ t_2,\omega - q\ t_1,\omega\right| + \frac{1}{\Gamma\alpha} \int_{t_1}^{t_2} \lambda(t) ds
\end{align*}
\]

\[
\begin{align*}
&\leq \left|q\ t_2,\omega - q\ t_1,\omega\right| + \frac{1}{\Gamma\alpha} \int_{t_1}^{t_2} \lambda(t) ds
\end{align*}
\]
Since \( q \) and \( p \) are continuous on the \( x_n \) for each \( \omega \in \Omega \) they are uniformly continuous. From (3.9) it follows that 
\[
\left| B \omega \ x_n \ t_2 - B \omega \ x_n \ t_1 \right| \to 0 \quad \text{as} \quad t_1 \to t_2.
\]

Hence \( B \omega \ x_n \) is an equicontinuous set in \( B \omega \ S \) for each \( \omega \in \Omega \). Now an application of Arzela-Ascoli theorem yield that \( B \omega \ S \) is compact for each \( \omega \in \Omega \) As a result \( B \omega \) is a compact random operator on \( S \).

Step IV: Next we show that \( A \omega \ B \omega \ x \in S \) for all \( x \in S \) is arbitrary, then 
\[
\left| A \omega \ x \ t \ B \omega \ x \ t \right| \leq |A \omega \ x \ t| \left| B \omega \ x \ t \right|
\]
\[
\leq \int t, x \ \theta \ t, \omega \ \left( q t, \omega \ + \frac{1}{\Gamma \alpha} \int_0^{t/\omega} g t, x \ \eta \ s \ \omega \ \omega \ ds \right) \phi \left( q t, \omega \ + \frac{h_{M \omega} t, \omega}{\Gamma \alpha+1} \right) \leq \left[ q t, \omega \ + \frac{h_{M \omega} t, \omega}{\Gamma \alpha+1} \right] \phi \left( \frac{a t}{\Gamma \alpha+1} \right)
\]
\[
\leq \phi \left( q t, \omega \ + \frac{h_{M \omega} t, \omega}{\Gamma \alpha+1} \right) \leq \phi \left( q t, \omega \ + \frac{a t}{\Gamma \alpha+1} \right)
\]
\[
\leq \phi \left( K_1 + K_2 \right) = r \quad 3.10
\]

for all \( t \in R_+ \). Taking the supremum over \( t \), we obtain \( \|A \omega x\| \leq r \) for all \( x \in S \). Hence hypothesis (c) of Theorem 2.4 holds.

Also we have
\[
M \omega = \|B \omega \ S\| = \sup \|B x\| : x \in S
\]
\[
= \sup \left\{ \sup_{t \geq 0} \left( q t, \omega \ + \frac{1}{\Gamma \alpha} \int_0^{t/\omega} g t, x \ \eta \ s \ \omega \ \omega \ ds \right) : x \in S \right\}
\]
\[
\leq \sup_{t \geq 0} \left[ q t, \omega \ + \frac{h_{M \omega} t, \omega}{\Gamma \alpha+1} \right] \leq K_1 + K_2 \quad 3.11
\]
and therefore \( M \phi_{\phi_t} \phi_{t} = \phi_{\phi_t} K_1 + K_2 < 1 \). Now we apply Theorem 2.4 to conclude that FRIE (1.1) has a solution on \( \mathbb{R}_+ \).

**Step VI:** Finally, we show the locally attractive of the solutions for FRIE (1.1). Let \( x \) and \( y \) be any two solutions of the FRIE (1.1) in \( S \) defined on \( \mathbb{R}_+ \). Then we have,

\[
|x(t) - y(t)| \leq \left| f(t, x(t), \theta_t, \omega) \left( q(t, \omega) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s, x(s), \eta_s, \omega)}{t-s^{1-\alpha}} ds \right) \right|
\]

\[
+ \left| f(t, y(t), \theta_t, \omega) \left( q(t, \omega) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s, y(s), \eta_s, \omega)}{t-s^{1-\alpha}} ds \right) \right|
\]

\[
\leq \left| f(t, x(t), \theta_t, \omega) \left( q(t, \omega) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s, x(s), \eta_s, \omega)}{t-s^{1-\alpha}} ds \right) \right|
\]

\[
+ \left| f(t, y(t), \theta_t, \omega) \left( q(t, \omega) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s, y(s), \eta_s, \omega)}{t-s^{1-\alpha}} ds \right) \right|
\]

\[
\leq \phi_{\omega} \left( |q(t, \omega)| + \frac{1}{\Gamma(\alpha)} \int_0^t h_{m, \omega}(t, \omega) \frac{1}{t-s^{1-\alpha}} ds \right) + \phi_{\omega} \left( |q(t, \omega)| + \frac{1}{\Gamma(\alpha)} \int_0^t h_{m, \omega}(t, \omega) \frac{1}{t-s^{1-\alpha}} ds \right)
\]

\[
\leq 2\phi_{\omega} \left( |q(t, \omega)| + \frac{1}{\Gamma(\alpha)} \int_0^t h_{m, \omega}(t, \omega) \frac{1}{t-s^{1-\alpha}} ds \right)
\]

\[
\leq 2\phi_{\omega} \left( |q(t, \omega)| + \frac{h_{m, \omega}(t, \omega)}{\Gamma(\alpha)} \int_0^t \frac{1}{t-s^{1-\alpha}} ds \right) \leq 2\phi_{\omega} \left( |q(t, \omega)| + \frac{\alpha t}{\Gamma(\alpha+1)} \right)
\]

\[
\leq 2\phi_{\omega} \left( |q(t, \omega)| + \frac{a t}{\Gamma(\alpha+1)} \right) 3.12
\]

for all \( t \in \mathbb{R}_+ \). Since \( \lim_{t \to \infty} q(t, \omega) = 0 \) and \( \lim_{t \to \infty} a t = 0 \), for \( \varepsilon > 0 \), there are real numbers \( T > 0 \) and \( T' > 0 \) such that \( |q(t, \omega)| < \frac{\varepsilon}{4\phi_{\omega}} \) for all \( t \geq T \) and \( a t \leq \frac{\Gamma(\alpha+1)\varepsilon}{4\phi_{\omega}} \) for all \( t \geq T' \). If we choose \( T'' = \max(T, T') \), then from the above inequality it follows that \( |x(t) - y(t)| \leq \varepsilon \) for all \( t \geq T'' \). Now, it is easy to prove that every solution of the FRIE (1.1) is asymptotic to the \( x(t) = 0 \) on \( \mathbb{R}_+ \). Consequently, the FRIE (1.1) has a solution and all the solutions are uniformly globally asymptotically attractive on \( \mathbb{R}_+ \) with the line \( x(t) = 0 \), \( t \in \mathbb{R}_+ \) as a common attractor to them. This completes the prof.

**Acknowledgement:** This article is a some output result of Minor Research Project funded by UGC, New Delhi, India.
References


