SOME TOPOLOGICAL STRUCTURES OF NORMED SPACE VALUED ORLICZ FUNCTION SPACE (l_{\infty}(U, V, ||.||, \Omega))

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Abstract: The aim of this work is to study new classes l_{\infty}(U, V, ||.||, \Omega) and l_s(U, V, ||.||, \Omega, \lambda) of normed space valued function space using Orlicz function \Omega as the generalizations of well known complex basic sequence space l_{\infty} studied in Functional Analysis. These results can be used for further generalization to investigate the various properties of existing Orlicz function spaces. Besides the investigation of linear topological structures of the class l_{\infty}(U, V, ||.||, \Omega) when topologized it with suitable natural norm, our primarily interest is to explore the various conditions pertaining to the inclusion of the class l_s(U, V, ||.||, \Omega, \lambda) in terms of different values of \lambda.

Key words: Normed space, Normal space, Orlicz function, Orlicz sequence space, Orlicz function space.

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1. INTRODUCTION

So far, a bulk number of research works have been made on various types of structures in normed space valued function spaces. Before proceeding with the main results, we recall some of the basic notations and definitions that are used in this paper.

Definition 1.1: A normed space (U, ||.||) is a linear space U together with the mapping ||.|| : U \rightarrow \mathbb{R},
(called norm on U) such that for all u, w \in U and \alpha \in \mathbb{C}, we have

N_1: \quad ||u|| \geq 0 and ||u|| = 0 if and only if u = 0 , where 0 is the zero element of U;

N_2: \quad ||\alpha u|| = ||\alpha|| ||u||; and

N_3: \quad ||u + w|| \leq ||u|| + ||w||.

Clearly by N_1 and N_2, algebraic operations of addition and scalar multiplication in the normed space U are continuous i.e., if (\{u_n\}) and (\{w_n\}) are sequences in the normed space U with u, w \in U such that u_n \rightarrow u, w_n \rightarrow w in U, and (\{\alpha_n\}) a sequence of scalars with \alpha \in \mathbb{C} such that \alpha_n \rightarrow \alpha in \mathbb{C} then

\quad u_n + w_n \rightarrow u + w and \alpha_n u_n \rightarrow \alpha u in U.

In fact, Maddox (1969), Pahari (2011), Srivastava and Pahari (2011) and many others have been introduced and studied the algebraic and topological properties of various function spaces in normed space. All these function spaces generalize and unify various existing basic sequence spaces studied in Functional Analysis.

Definition 1.2: Let V be a normed space and

\quad S(V) = \{ \phi : U \rightarrow V \}

be the classes of V-valued functions. Then S(V) is called solid (or normal) if \phi \epsilon S(V) and scalars \alpha(u), u \in U such that |\alpha(u)| \leq 1 , u \in U implies

\quad \alpha(u) \phi(u) \in S(V).

Definition 1.3: An Orlicz function is a function \Omega : [0,\infty) \rightarrow [0,\infty) which is continuous, non decreasing and convex with

\quad \Omega (0) = 0, \quad \Omega (u) > 0 for u > 0, \quad and \quad \Omega (u) \rightarrow \infty \quad as \quad u \rightarrow \infty.

An Orlicz function \Omega can be represented in the following integral form
\[ \Omega(u) = \int_0^u q(t) \, dt \]

where \( q \), known as the kernel of \( \Omega \), is right-differentiable for \( t \geq 0 \), \( q(0) = 0 \), \( q(t) > 0 \) for \( t > 0 \), \( q \) is non decreasing, and \( q(t) \to \infty \) as \( t \to \infty \) (see, Krasnosel’skii and Rutickii, 1961).

**Definition 1.4:** An Orlicz function \( \Omega \) is said to satisfy \( \Delta_2 \)-condition for all values of \( t \), if there exists a constant \( K > 0 \) such that

\[ \Omega(2t) \leq K \Omega(t) \]

for all \( t \geq 0 \).

The \( \Delta_2 \)-condition is equivalent to the satisfaction of the inequality

\[ \Omega(Lt) \leq KL \Omega(t) \]

for all values of \( t \) for which \( L > 1 \), (see, Krasnosel’skii and Rutickii, 1961).

**Definition 1.5:** Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct the sequence space

\[ l_\Omega = \inf \left\{ u = (u_k) : \sum_{k=1}^{\infty} \Omega \left( \frac{|u_k|}{r} \right) < \infty \text{ for some } r > 0 \right\} \]

of scalars \( (u_k) \), which forms a Banach space with Luxemburg norm defined by

\[ \| u \|_\Omega = \inf \left\{ r > 0 : \sum_{k=1}^{\infty} \Omega \left( \frac{|u_k|}{r} \right) \leq 1 \right\}. \]

The space \( l_\Omega \) is called an Orlicz sequence space and is closely related to the space \( l_p \) with

\[ \Omega(u) = u^p, (1 \leq p < \infty). \]

They have very rich topological and geometrical properties that do not occur in ordinary \( l_p \) space.

Kamthan and Gupta (1981), Rao and Ren (1991), Parashar and Choudhary (1994), Ghosh and Srivastava (1999), Bhardwaj and Bala (2007), Khan (2008), Basariv a (2009), Kolk (2011), Pahari (2011), Srivastava and Pahari (2011) and many others have been introduced and studied the algebraic and topological properties of various sequence and function spaces using Orlicz function as the generalizations of various well known sequence spaces and function spaces.

2. The Classes \( l_\omega (U, V, \| \cdot \|, \Omega) \) and \( l_\omega (U, V, \| \cdot \|, \Omega, \lambda) \)

Let \( U \) be an arbitrary non-empty set (not necessarily countable) and \( \mathcal{F}(U) \) be the collection of all finite subsets of \( U \) directed by inclusion relation. Let \( (V, \| \cdot \|) \) be a Banach space over the field of complex number \( C \). We shall write \( \lambda, \mu \) for functions on \( U \to C \setminus \{0\} \) and the collection of all such functions will be denoted by \( s(U, C \setminus \{0\}) \).

We shall also frequently use the notation

\[ q(u) = \left| \frac{\lambda(u)}{\mu(u)} \right| \]

We now introduce the following new classes of Banach space \( V \)-valued functions using Orlicz function \( \Omega \).

\[ l_\omega (U, V, \| \cdot \|, \Omega) = \{ \phi : U \to V : \sup_{u \in U} \Omega \left( \frac{\| \phi(u) \|}{r} \right) < \infty \text{ for some } r > 0 \}; \quad \ldots(2.1) \]

and

\[ l_\omega (U, V, \| \cdot \|, \Omega, \lambda) = \{ \phi : U \to V : \sup_{u \in U} \Omega \left( \frac{\| \lambda(u) \phi(u) \|}{r} \right) < \infty \text{ for some } r > 0 \} \ldots(2.2) \]

Moreover, we define the subclass \( \bar{l}_\omega (U, V, \| \cdot \|, \Omega) \) of \( l_\omega (U, V, \| \cdot \|, \Omega) \) as

\[ \bar{l}_\omega (U, V, \| \cdot \|, \Omega) = \{ \phi : U \to V : \sup_{u \in U} \Omega \left( \frac{\| \phi(u) \|}{r} \right) < \infty \text{ for every } r > 0 \} \ldots(2.3) \]

Further when \( \lambda : U \to C \setminus \{0\} \) is a function such that \( \lambda(u) = 1 \) for all \( u \), then \( l_\omega (U, V, \| \cdot \|, \Omega, \lambda) \) will be denoted by \( l_\omega (U, V, \| \cdot \|, \Omega) \).

3. Linear Topological Structure of \( l_\omega (U, V, \| \cdot \|, \Omega) \)

As far as linear space structures of the class \( l_\omega (U, V, \| \cdot \|, \Omega) \) over the field \( C \) are concerned, we shall take point-wise vector operations, i.e., for any \( \phi, \psi \in l_\omega (U, V, \| \cdot \|, \Omega) \), we have

\[ (\phi + \psi)(u) = \phi(u) + \psi(u), u \in U \]

and

\[ (\alpha \phi)(u) = \alpha \phi(u), u \in U, \alpha \in C. \]

Moreover, we shall denote zero element of the space by \( 0 \) by which we mean the function
Theorem 3.1: If \( \Omega \) satisfies the \( \Delta_2 \)–condition then we have

\[
\mathbb{L}_\omega (U, V, \| \cdot \|, \Omega) = \mathcal{I}_\omega (U, V, \| \cdot \|, \Omega).
\]

Proof:

Clearly \( \mathbb{L}_\omega (U, V, \| \cdot \|, \Omega) \supseteq \mathcal{I}_\omega (U, V, \| \cdot \|, \Omega) \).

Hence to prove the assertion it is sufficient to show that \( \mathbb{L}_\omega (U, V, \| \cdot \|, \Omega) \) is a subset of \( \mathcal{I}_\omega (U, V, \| \cdot \|, \Omega) \).

Suppose \( \phi \in \mathbb{L}_\omega (U, V, \| \cdot \|, \Omega) \) then for some \( r > 0 \),

\[
\sup_{u \in U} \Omega \left( \frac{\| \phi(u) \|}{r} \right) < \infty.
\]

Let \( \eta > 0 \). If \( r \leq \eta \) then clearly

\[
\sup_{u \in U} \Omega \left( \frac{\| \phi(u) \|}{\eta} \right) < \sup_{u \in U} \Omega \left( \frac{\| \phi(u) \|}{r} \right) < \infty.
\]

Let now \( \eta < r \) and put \( l = \frac{r}{\eta} > 1 \). Since, \( \Omega \) satisfies the \( \Delta_2 \)–condition, there exists a constant \( K \) such that

\[
\Omega \left( \frac{\| \phi(u) \|}{\eta} \right) = \Omega \left( \frac{r}{\eta} \cdot \frac{\| \phi(u) \|}{r} \right) \leq K \frac{r}{\eta} \Omega \left( \frac{\| \phi(u) \|}{r} \right) < \infty
\]

for each \( u \in U \), which implies that

\[
\sup_{u \in U} \Omega \left( \frac{\| \phi(u) \|}{\eta} \right) < \infty \text{ for every } \eta > 0.
\]

This shows that

\[
\mathbb{L}_\omega (U, V, \| \cdot \|, \Omega) \subseteq \mathcal{I}_\omega (U, V, \| \cdot \|, \Omega)
\]

and hence

\[
\mathbb{L}_\omega (U, V, \| \cdot \|, \Omega) = \mathcal{I}_\omega (U, V, \| \cdot \|, \Omega).
\]

This completes the proof.

Theorem 3.2: The class \( \mathbb{L}_\omega (U, V, \| \cdot \|, \Omega) \) is normal.

Proof:

Let \( \phi \in \mathbb{L}_\omega (U, V, \| \cdot \|, \Omega) \) and \( r > 0 \) be associated with \( \phi \).

Then we have

\[
\sup_{u \in U} \Omega \left( \frac{\| \phi(u) \|}{r} \right) < \infty.
\]

Now, if we take scalars \( \alpha(u), u \in U \) such that \( |\alpha(u)| \leq 1 \), then

\[
\sup_{u \in U} \Omega \left( \frac{\| \alpha(u) \phi(u) \|}{r} \right) \leq \sup_{u \in U} \Omega \left( \frac{|\alpha(u)|}{r} \| \phi(u) \| \right) \leq \sup_{u \in U} \Omega \left( \frac{\| \phi(u) \|}{r} \right) < \infty.
\]

This shows that \( \alpha \phi \in \mathbb{L}_\omega (U, V, \| \cdot \|, \Omega) \) and hence \( \mathbb{L}_\omega (U, V, \| \cdot \|, \Omega) \) is normal.

Theorem 3.3: The class \( \mathbb{L}_\omega (U, V, \| \cdot \|, \Omega) \) is a linear space over the filed \( C \) with respect to the point wise vector operations.

Proof:

Let us suppose that \( \phi, \psi \in \mathbb{L}_\omega (U, V, \| \cdot \|, \Omega) \), \( r_1 > 0 \) and \( r_2 > 0 \) associated with \( \phi \) and \( \psi \) respectively and \( \alpha, \beta \in C \). Then there exist \( J_1, J_2 \in \mathcal{F}(U) \) such that

\[
\sup_{u \in U} \Omega \left( \frac{\| \phi(u) \|}{r_1} \right) < \infty, \text{ for all } u \in U / J_1,
\]

and

\[
\sup_{u \in U} \Omega \left( \frac{\| \psi(u) \|}{r_2} \right) < \infty, \text{ for all } u \in U / J_2.
\]
Let
\[ r_3 = \max (2|\alpha| r_1, 2|\beta| r_2). \]

Since \( \Omega \) is non-decreasing and convex so for \( u \in U / (J_1 \cup J_2) \), we have
\[
\sup_{u} \Omega \left( \frac{\| \alpha \phi(u) + \beta \psi(u) \|}{r_3} \right) \leq \sup_{u} \Omega \left( \frac{\| \alpha \phi(u) \|}{r_3} + \frac{\| \beta \psi(u) \|}{r_3} \right) \\
\leq \frac{1}{2} \sup_{u} \Omega \left( \frac{\| \phi(u) \|}{r_1} \right) + \frac{1}{2} \sup_{u} \Omega \left( \frac{\| \psi(u) \|}{r_2} \right) \\
< \infty, \text{ for all } u \in U / (J_1 \cup J_2),
\]
which shows that
\[ \alpha \phi + \beta \psi \in l_\infty (U, V, \| \cdot \|, \Omega). \]

Hence, \( l_\infty (U, V, \| \cdot \|, \Omega) \) forms a linear space.

**Theorem 3.4:** The linear space \( l_\infty (U, V, \| \cdot \|, \Omega) \) forms a normed space with respect to norm
\[
\| \phi \|_\infty = \inf \{ \sup_{u} \Omega \left( \frac{\| \phi(u) \|}{r} \right) : r > 0 \} \leq 1. \tag{3.1}
\]

**Proof:**

**For \( N_1: \** Clearly, \( \| \phi \|_\infty \geq 0 \), for all \( \phi \in l_\infty (U, V, \| \cdot \|, \Omega)\).

Since \( \Omega (0) = 0 \), therefore for \( \phi = 0 \) we easily get
\[ \| \phi \|_\infty = 0. \]

Conversely suppose that \( \| \phi \|_\infty = 0 \), i.e.
\[ \inf \{ \sup_{u} \Omega \left( \frac{\| \phi(u) \|}{r} \right) : r > 0 \} = 0. \]

This implies that for given \( \epsilon > 0 \) there exists some \( r_\epsilon \) \((0 < r_\epsilon < \epsilon)\), such that
\[ \sup_{u} \Omega \left( \frac{\| \phi(u) \|}{r_\epsilon} \right) \leq 1. \]

Thus,
\[ \sup_{u} \Omega \left( \frac{\| \phi(u) \|}{r_\epsilon} \right) \leq \sup_{u} \Omega \left( \frac{\| \phi(u) \|}{r} \right) \leq 1. \]

Suppose \( \phi(u) \neq 0 \) for some \( u \in U \). Then clearly \( \epsilon \to 0 \) implies that
\[ \frac{\| \phi(u) \|}{\epsilon} \to \infty \]
which contradicts that
\[ \sup_{u} \Omega \left( \frac{\| \phi(u) \|}{\epsilon} \right) \leq 1. \]

Hence \( \phi(u) = 0 \) for each \( u \in U \) and so \( \phi = 0 \).

**For \( N_2: \** Let \( r_1 > 0 \) and \( r_2 > 0 \) be such that
\[ \sup_{u} \Omega \left( \frac{\| \phi(u) \|}{r_1} \right) \leq 1, \text{ for all } u \in U \]
and
\[ \sup_{u} \Omega \left( \frac{\| \psi(u) \|}{r_2} \right) \leq 1, \text{ for all } u \in U. \]

Then for
\[ r_3 = r_1 + r_2 \]
we have
\[ \Omega \left( \frac{\| \phi(u) + \psi(u) \|}{r_3} \right) \leq \Omega \left( \frac{\| \phi(u) \| + \| \psi(u) \|}{r_1 + r_2} \right) \]
\[ \leq \frac{r_1}{r_1 + r_2} \Omega \left( \frac{\| \phi(u) \|}{r_1} \right) + \frac{r_2}{r_1 + r_2} \Omega \left( \frac{\| \psi(u) \|}{r_2} \right) \]
which implies that
\[ \sup_{u} \Omega \left( \frac{\| \phi(u) + \psi(u) \|}{r_3} \right) \leq \frac{r_1}{r_1 + r_2} \sup_{u} \Omega \left( \frac{\| \phi(u) \|}{r_1} \right) + \frac{r_2}{r_1 + r_2} \sup_{u} \Omega \left( \frac{\| \psi(u) \|}{r_2} \right) \]
\[ \leq 1. \]

Thus
Theorem 3.

Let

\[
\text{sup}_{u \in U} \Omega \left( \frac{\| \phi(u) + \psi(u) \|}{r} \right) \leq 1
\]

implies that

\[
\text{inf} \left\{ r > 0: \text{sup}_{u \in U} \Omega \left( \frac{\| \phi(u) + \psi(u) \|}{r} \right) \leq 1 \right\}
\]

\[
\leq \text{inf} \left\{ r_1 > 0: \text{sup}_{u \in U} \Omega \left( \frac{\| \phi(u) \|}{r_1} \right) \leq 1 \right\} + \text{inf} \left\{ r_2 > 0: \text{sup}_{u \in U} \Omega \left( \frac{\| \psi(u) \|}{r_2} \right) \leq 1 \right\},
\]

and hence by (3.1) we get

\[\| \phi + \psi \| \leq \| \phi \| + \| \psi \|\].

For \( N_3 \): If \( \alpha = 0 \), then obviously

\[\| \alpha \phi \| \leq |\alpha| \| \phi \|\].

Now, suppose that \( \alpha \neq 0 \)

\[\| \alpha \phi \| \leq \text{inf} \left\{ r > 0: \text{sup}_{u \in U} \Omega \left( \frac{\| \alpha \phi(u) \|}{r} \right) \leq 1 \right\} = |\alpha| \| \phi \|_{s}.
\]

This shows that \( l_{\infty} (U, V, \| \cdot \|, \Omega) \) is a normed space with respect to the norm \( \| \cdot \|_{s}\).

This completes the proof.

Theorem 3.5: For any \( \lambda, \mu \in s (U, C \setminus \{0\}) \),

\[l_{\infty} (U, V, \| \cdot \|, \Omega, \lambda) \subseteq l_{\infty} (U, V, \| \cdot \|, \Omega, \mu)\] if \( \lim \inf_{u} t(u) > 0 \).

Proof:

Assume that \( \lim \inf_{u} t(u) > 0 \). Then there exists positive constant \( m \) such that

\[m \mu(u) < \| \lambda(u) \|_{s},\]

for all but finitely many \( u \in U \).

Let \( \phi \in l_{\infty} (U, V, \| \cdot \|, \Omega, \lambda) \) and \( r_1 > 0 \) is associated with \( \phi \), so that

\[\text{sup}_{u \in U} \Omega \left( \frac{\| \lambda(u) \|_{\phi(u)} \|}{r_1} \right) < \infty.
\]

Let us choose \( r \) such that \( r_1 < m r \).

Then for such \( r \), using non decreasing property of \( \Omega \), we have

\[\text{sup}_{u \in U} \Omega \left( \frac{\| \mu(u) \|_{\phi(u)} \|}{r} \right) = \text{sup}_{u \in U} \Omega \left( \frac{\| \mu(u) \|_{\phi(u)} \|}{r} \right) \leq \text{sup}_{u \in U} \Omega \left( \frac{\| \lambda(u) \|_{\phi(u)} \|}{r_1} \right) < \infty.
\]

This shows that \( \phi \in l_{\infty} (U, V, \| \cdot \|, \Omega, \mu) \) and hence

\[l_{\infty} (U, V, \| \cdot \|, \Omega, \lambda) \subseteq l_{\infty} (U, V, \| \cdot \|, \Omega, \mu).
\]

Theorem 3.6: For any \( \lambda, \mu \in s (U, C \setminus \{0\}) \), if

\[l_{\infty} (U, V, \| \cdot \|, \Omega, \lambda) \subset l_{\infty} (U, V, \| \cdot \|, \Omega, \mu)\] then \( \lim \inf_{u} t(u) > 0 \).

Proof:

Assume that

\[l_{\infty} (U, V, \| \cdot \|, \Omega, \lambda) \subset l_{\infty} (U, V, \| \cdot \|, \Omega, \mu)\]

but

\[\lim \inf_{u} t(u) = 0.
\]

Then there exists a sequence \( (u_k) \) in \( U \) of distinct points such that for each \( k \geq 1 \), we have

\[k \| \lambda(u_k) \| < \| \mu(u_k) \| \]

… (3.2)

We now choose \( v \in V \) and \( \| v \| = 1 \) and define \( \phi : U \rightarrow V \) by

\[\phi(u) = \left( \lambda(u_k) \right)^{-1} v, \text{ for } u = u_k, k = 1, 2, 3, \ldots \text{, and } 0, \text{ otherwise.}\]

\[\phi(u) = \left( \lambda(u_k) \right)^{-1} v, \text{ for } u = u_k, k = 1, 2, 3, \ldots \text{, and } 0, \text{ otherwise.}\]

… (3.3)

Let \( r > 0 \). Then we have
Then we can find a sequence \( (u_k) \) of distinct points in \( U \) such that for each \( k \geq 1 \),
\[
\frac{\| \lambda(u) \phi(u) \|}{r} = \frac{\| \mu(u) \phi(u) \|}{r} = \Omega \left( \frac{1}{r} \right).
\]

This clearly shows that \( \phi \in l_v(U, V, \| \cdot \|, \Omega, \lambda) \).

But on the other hand, in view of (3.2) and (3.3), we have
\[
\sup_{u \in U} \Omega \left( \frac{\| \mu(u) \phi(u) \|}{r} \right) = \sup_{k \geq 1} \Omega \left( \frac{\| \mu(u_k) \phi(u_k) \|}{r} \right).
\]

\[
= \Omega \left( \frac{1}{r} \right) \left( \sup_{k \geq 1} \frac{\| \mu(u_k) \|}{\lambda(u_k)} \| u_k \| \right).
\]

\[
\geq \sup_{k \geq 1} \Omega \left( \frac{k}{\rho} \right) = \infty.
\]

This shows that \( \phi \notin l_\varepsilon(U, V, \| \cdot \|, \Omega, \mu) \), a contradiction.

This completes the proof.

Combining theorem 3.5 and 3.6, we have

**Theorem 3.7:** For any \( \lambda, \mu \in s(U, C \setminus \{0\}) \),
\[
l_\varepsilon(U, V, \| \cdot \|, \Omega, \lambda) \subset l_\varepsilon(U, V, \| \cdot \|, \Omega, \mu) \text{ if and only if } \liminf u \varepsilon u > 0.
\]

**Theorem 3.8:** Let \( \lambda, \mu \in s(U, C \setminus \{0\}) \),
\[
l_\varepsilon(U, V, \| \cdot \|, \Omega, \mu) \subset l_\varepsilon(U, V, \| \cdot \|, \Omega, \lambda) \text{ if } \limsup u \varepsilon u < \infty.
\]

**Proof:**

Assume that \( \limsup u \varepsilon u < \infty \). Then there exists \( d > 0 \) such that
\[
|\lambda(u)| < d |\mu(u)|
\]

for all but finitely many \( u \in U \).

Let \( \phi \in l_\varepsilon(U, V, \| \cdot \|, \Omega, \mu) \) and \( r_1 > 0 \) is associated with \( \phi \).

Then
\[
\sup_{u \in U} \Omega \left( \frac{\| \mu(u) \phi(u) \|}{r_1} \right) < \infty.
\]

Let us choose \( r > 0 \) such that \( d r_1 \leq r \).

Then for such \( r \), using non-decreasing property of \( \Omega \) we have
\[
\sup_{u \in U} \Omega \left( \frac{\| \lambda(u) \phi(u) \|}{r} \right) \leq \sup_{u \in U} \Omega \left( \frac{d |\mu(u)| \| \phi(u) \|}{r} \right) \leq \sup_{u \in U} \Omega \left( \frac{\| \mu(u) \phi(u) \|}{r_1} \right) < \infty.
\]

This shows that \( \phi \in l_\varepsilon(U, V, \| \cdot \|, \Omega, \lambda) \)

and hence
\[
l_\varepsilon(U, V, \| \cdot \|, \Omega, \mu) \subset l_\varepsilon(U, V, \| \cdot \|, \Omega, \lambda).
\]

**Theorem 3.9:** Let \( \lambda, \mu \in s(U, C \setminus \{0\}) \), and
\[
l_\varepsilon(U, V, \| \cdot \|, \Omega, \mu) \subset l_\varepsilon(U, V, \| \cdot \|, \Omega, \lambda) \text{ then } \limsup u \varepsilon u < \infty.
\]

**Proof:**

Assume that
\[
l_\varepsilon(U, V, \| \cdot \|, \Omega, \mu) \subset l_\varepsilon(U, V, \| \cdot \|, \Omega, \lambda)
\]

But
\[
\limsup u \varepsilon u = \infty.
\]

Then we can find a sequence \( (u_k) \) of distinct points in \( U \) such that for each \( k \geq 1 \),
\[
|\lambda(u_k)| > k |\mu(u_k)| \quad \ldots \quad (3.4)
\]

We now choose \( v \in V \) and \( \| v \| = 1 \) and define \( \phi : U \rightarrow V \) by
\[ \phi(u) = \begin{cases} \frac{1}{(\mu(u_k))^{\frac{1}{k}}} v, & \text{for } u = u_k, k = 1, 2, 3, \ldots, \text{ and} \\ 0, & \text{otherwise.} \end{cases} \quad (3.5) \]

Let \( r > 0 \).

Then we have
\[
\sup_{u \in U} \Omega \left( \frac{\| \mu(u) \phi(u) \|}{r} \right) = \sup_{k \geq 1} \Omega \left( \frac{\| \mu(u_k) \phi(u_k) \|}{r} \right) = \Omega \left( \frac{1}{r} \right).
\]

This shows that \( \phi \in \ell_\infty(U, V, \| \cdot \|, \Omega, \mu) \).

But in view of (3.4) and (3.5), we have
\[
\sup_{u \in U} \Omega \left( \frac{\| \lambda(u) \phi(u) \|}{r} \right) = \sup_{k \geq 1} \Omega \left( \frac{\| \lambda(u_k) \phi(u_k) \|}{r} \right) = \sup_{k \geq 1} \Omega \left( \frac{\| u \|}{r} \right) = \infty,
\]

implies that
\( \phi \notin \ell_\infty(U, V, \| \cdot \|, \Omega, \lambda, p) \).

This leads to a contradiction and completes the proof.

Combining theorem 3.8 and 3.9, we have

**Theorem 3.10:** Let \( \lambda, \mu \in s(U, C \setminus \{0\}) \),
\[
\ell_\infty(U, V, \| \cdot \|, \| \Omega, \mu \| \subset \ell_\infty(U, V, \| \cdot \|, \Omega, \lambda) \text{ if and only if } \limsup \ell(t(u)) < \infty.
\]

When the Theorems 3.7 and 3.10 are combined, we get :

**Theorem 3.11:** If \( \lambda, \mu \in s(X, C \setminus \{0\}) \), then
\[
\ell_\infty(U, V, \| \cdot \|, \| \Omega, \lambda \|) = \ell_\infty(U, V, \| \cdot \|, \| \Omega, \mu \|) \text{ if and only if } \\
0 < \liminf \ell(t(u)) \leq \limsup \ell(t(u)) < \infty.
\]

**Corollary 3.12:** Let \( \lambda \in s(U, C \setminus \{0\}) \). Then

(i) \( \ell_\infty(U, V, \| \cdot \|, \lambda \| \Omega, \lambda \| \subset \ell_\infty(U, V, \| \cdot \|, \mu) \)

if and only if \( \liminf |\lambda(u)| > 0 \);

(ii) \( \ell_\infty(U, V, \| \cdot \|, \Omega, \lambda) \subset \ell_\infty(U, V, \| \cdot \|, \| \lambda(u) \| \}

if and only if \( \limsup |\lambda(u)| < \infty \); and

(iii) \( \ell_\infty(U, V, \| \cdot \|, \| \Omega \|, \lambda) = \ell_\infty(U, E, V, \| \cdot \|, \| \lambda(u) \| \}

if and only if
\[
0 < \liminf |\lambda(u)| \leq \limsup |\lambda(u)| < \infty.
\]

**Proof:**

By considering the function \( \mu \) on \( U \) such that \( \mu(u) = 1 \) for all \( u \in U \) in Theorems 3.7, 3.10 and 3.11, one can easily obtain the assertions (i), (ii) and (iii) respectively.

**Conclusion:**

In this paper, we have established some of the results that characterize the linear topological structures of the classes \( \ell_\infty(U, V, \| \cdot \|, \Omega) \) and \( \ell_\infty(U, V, \| \cdot \|, \| \Omega, \lambda \| \) of normed space valued Orlicz function space studied in Functional Analysis. In fact, these results can be used for further generalization to investigate the various properties of existing Orlicz function spaces.

**References**


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